Algebraic structures connected with pairs of compatible associative algebras

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Abstract

We study associative multiplications in semi-simple associative algebras over $\mathbb{C}$ compatible with the usual one or, in other words, linear deformations of semi-simple associative algebras over $\mathbb{C}$. It turns out that these deformations are in one-to-one correspondence with representations of certain algebraic structures, which we call M-structures in the matrix case and PM-structures in the case of direct sums of several matrix algebras. We also investigate various properties of PM-structures, provide numerous examples and describe an important class of PM-structures. The classification of these PM-structures naturally leads to affine Dynkin diagrams of A, D, E-type.

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Introduction

Two associative algebras with multiplications $\star$ and $\circ$ defined on the same finite dimensional vector space $\mathbf{V}$ are said to be *compatible* if the multiplication

$$a \bullet b = a \star b + \lambda\, a \circ b \tag{0.1}$$

is associative for any constant λ. The multiplication $\bullet$ can be regarded as a deformation of the multiplication $\star$ linear in parameter λ.

The description of pairs of compatible associative products seems to be an interesting mathematical problem on its own. Moreover, the approach to integrable systems based on the concept of compatible Poisson structures via Lenard-Magri scheme [1] provides further motivation for investigation of compatible associative multiplications.

Recall that two Poisson brackets are said to be *compatible* if any linear combination of these brackets is a Poisson bracket. It is well-known that the formula $\{x_i,\, x_j\} = c_{ij}^k x_k, \quad i, j = 1, \dots, N$ defines a linear Poisson structure iff c_{ij}^k are structural constants of a Lie algebra. The compatibility of two such structures is equivalent to the compatibility of the corresponding Lie brackets. Various applications of compatible Lie brackets in the integrability theory can be found in [3, 4, 5, 6, 2].

Suppose now that we have two compatible associative algebras with multiplications $\star$ and $\circ$ defined on the same finite dimensional vector space $\mathbf{V}$. We can construct immediately two compatible Lie brackets by the usual formulas $[a, b]_1 = a \star b - b \star a$ and $[a, b]_2 = a \circ b - b \circ a$ and hence, two compatible linear Poisson structures.

Moreover, for any $n \in \mathbb{N}$ we can construct two compatible associative algebras in the space $Mat_n(\mathbf{V})$, which is the space of $n \times n$ matrices with entries from $\mathbf{V}$. Therefore, for each n we have a pair of compatible Poisson structures in the linear space of dimension $n^2 \dim \mathbf{V}$. Note that even if both associative algebras on $\mathbf{V}$ are commutative we have nontrivial Poisson structures on the space $Mat_n(\mathbf{V})$ for $n > 1$. In terms of coordinates, if $\{e_i, i = 1, ...N\}$ is a basis of $\mathbf{V}$ and $e_i \star e_j = p_{i,j}^k e_k, \quad e_i \circ e_j = q_{i,j}^k e_k$, then for each n we have two compatible Poisson structures given, in coordinates $\{f_{i,l,m}, \quad i = 1, ..., N, \quad l, m = 1, ...n\}$, by the formulas

$$\{f_{i,l_1,m_1}, f_{j,l_2,m_2}\}_1 = \delta_{m_1,l_2} p_{i,j}^k f_{k,l_1,m_2} - \delta_{m_2,l_1} p_{j,i}^k f_{k,l_2,m_1}$$

and

$$\{f_{i,l_1,m_1}, f_{j,l_2,m_2}\}_2 = \delta_{m_1,l_2} q_{i,j}^k f_{k,l_1,m_2} - \delta_{m_2,l_1} q_{j,i}^k f_{k,l_2,m_1}.$$

Note that these Poisson structures are invariant with respect to the action of the group $GL_n(\mathbb{C})$ on the space $Mat_n(\mathbf{V})$ by conjugations. Therefore, for any two functions invariant with respect to this action their Poisson bracket is also invariant. Since any invariant function can be written in terms of traces of matrix polynomials, we see that a bracket of two traces can be also written in terms of traces. This leads us to bi-hamiltonian structures for the so-called nonabelian integrable systems in the sense of [9].

Another motivation for the study of compatible associative algebras can be found in [12].

In this paper we assume that the associative algebra over field $\mathbb{C}$ with multiplication $\star$ is semi-simple. In other words, this algebra is a direct sum of matrix algebras over $\mathbb{C}$ [8]. It turns out that in this case multiplications $\circ$ compatible with $\star$ are in one-to-one correspondence to representations of special infinite-dimensional associative algebras. The simplest finite-dimensional version of such an algebra can be described as follows. Let $\mathcal{A}$ and $\mathcal{B}$ be associative algebras of the same dimension p with bases $A_1, \dots, A_p$ and $B^1, \dots, B^p$ and structural constants $\phi^i_{j,k}$ and $\psi^{\alpha,\beta}_{\gamma}$, correspondingly. Suppose that the structural constants satisfy the following identities:

$$\phi^s_{j,k}\psi^{l,i}_s = \phi^l_{s,k}\psi^{s,i}_j + \phi^i_{j,s}\psi^{l,s}_k, \qquad 1 \le i, j, k, l \le p.$$

Then the algebra of dimension $2p + p^2$ with the basis A_i, B^j, A_iB^j and relations

$$B^iA_j = \psi^{k,i}_j A_k + \phi^i_{j,k}B^k$$

is associative.

An invariant description of such a construction can be given as follows. Suppose that we have two associative algebras $\mathcal{A}$ and $\mathcal{B}$, a non-degenerate pairing $\mathcal{A} \times \mathcal{B} \to \mathbb{C}$ and structures of right $\mathcal{A}$-module and left $\mathcal{B}$-module on the space $\mathcal{A} \oplus \mathcal{B}$ commuting with each other. Assume also that $\mathcal{A}$ acts in this module by right multiplication on itself and $\mathcal{B}$ acts by left multiplication on itself. Extend our pairing to the space $\mathcal{A} \oplus \mathcal{B}$ by the formulas $(a_1, a_2) = (b_1, b_2) = 0$ for $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and assume that it is invariant under the action of $\mathcal{A}$ and $\mathcal{B}$: $(va_1, a_2) = (v, a_1a_2)$ and $(b_1, b_2v) = (b_1b_2, v)$ for $v \in \mathcal{A} \oplus \mathcal{B}$. In this situation one can define a natural structure of an associative algebra on the space $\mathcal{A} \oplus \mathcal{B} \oplus (\mathcal{A} \otimes \mathcal{B})$ compatible with our module structures. This means that the action of $\mathcal{A}$ on the algebra by right multiplication restricted to $\mathcal{A} \oplus \mathcal{B}$ coincides with the module action and the same property is valid for the action of $\mathcal{B}$ by left multiplication.

Algebras considered in this paper are more complicated. Namely, algebras $\mathcal{A}$ and $\mathcal{B}$ have common unity. Instead of their direct sum, we construct a linear space of the same dimension but with one-dimensional "defect": $\mathcal{A}$ and $\mathcal{B}$ are intersected by the linear span of the unity but we add one more element, which in some sense dual to the unity. We assume the existence of a non-degenerate pairing and structures of a right $\mathcal{A}$-module and a left $\mathcal{B}$-module on this space with properties similar to described above. A linear space with these structures and the corresponding associative algebra are called *M-structure* and *M-algebra*. In turns out that M-algebra is infinite-dimensional over $\mathbb{C}$ but finite dimensional over the subalgebra generated by a special central element K.

The main result of this paper is the following: there is a one-to-one correspondence between n-dimensional representations (that should be non-degenerate in some sense) of M-algebras and associative products in Mat_n compatible with the usual matrix product. In other words, to find all associative products in matrix algebras compatible with the usual one, we should describe M-structures and for each M-structure classify finite-dimensional representations of the corresponding M-algebra.

To describe the compatible products for the algebra $Mat_{n_1} \oplus ... \oplus Mat_{n_m}$ we introduce PM-*algebras*, which are generalizations of M-algebras. Roughly speaking, a PM-algebra looks like the algebra of $m \times m$ matrices with entries being elements of some M-algebra.

This paper is organized as follows. In Section **1**, we collect some general facts about compatible associative multiplications. The first result of that Section is standard and based on the deformation theory of associative algebras. Namely, we show that if the algebra with respect to multiplication $\star$ is rigid (which holds in semi-simple case), then there exists a linear operator $R : \mathbf{V} \to \mathbf{V}$ such that the multiplication $\circ$ is of the form

$$X \circ Y = R(X) \star Y + X \star R(Y) - R(X \star Y). \tag{0.2}$$

We also provide several examples of compatible multiplications. At the end of Section **1** we give a construction of $m+1$ pairwise compatible associative multiplications on the space $\mathbf{V} \otimes F_m$ provided that we have two compatible associative multiplications on the space $\mathbf{V}$. Here F_m is the space of polynomials in one variable of degree less then m.

In Section **2**, we consider multiplications compatible with the standard matrix product in Mat_n. In Subsection **2.1** we study admissible operators R written in the form

$$R(x) = a_1\, x\, b^1 + ... + a_p\, x\, b^p + c\, x \tag{0.3}$$

with p being smallest possible. It turns out that $a_1, ..., a_p, b^1, ..., b^p, c$ should be generators of a representation of an M-structure. In Subsection **2.2**, we propose an invariant definition of M-structures and M-algebras and study their properties. In Subsection **2.3** we describe M-algebras in the special case when the algebra $\mathcal{A}$ is commutative semi-simple (that is, isomorphic to $\mathbb{C} \oplus ... \oplus \mathbb{C}$).

Section **3** is devoted to a generalization of the results from the previous section to the case of the algebra $Mat_{n_1} \oplus ... \oplus Mat_{n_m}$. All results and proofs are similar to the ones from Section **2**. In Subsection **3.1** we study possible operators R, and in Subsection **3.2** give an invariant definition of the corresponding algebraic structures.

In Section **4** we describe all PM-structures with semi-simple algebras $\mathcal{A}$ and $\mathcal{B}$. It turns out that such PM-structures are related to Cartan matrices of affine Dynkin diagrams of the $\tilde{A}_{2k-1}$, $\tilde{D}_k$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$-type.

In Conclusion we discuss some implications of our results and possible directions of further research.

1 Compatible associative multiplications

Suppose that we have an associative multiplication $\star$ defined on a finite dimensional vector space $\mathbf{V}$ such that $\mathbf{V}$ is a semi-simple algebra with respect to this multiplication. The following classification problem arises: to describe all possible associative multiplications $\circ$ on a vector

space $\mathbf{V}$, compatible with a given semi-simple multiplication $\star$. Since any semi-simple associative algebra is rigid, the multiplication (0.1) is isomorphic to $\star$ for almost all values of the parameter λ. This means that there exists a formal series of the form

$$A_\lambda = 1 + R\,\lambda + S\,\lambda^2 + \cdots, \tag{1.4}$$

where the coefficients are linear operators on $\mathbf{V}$, such that

$$A_\lambda^{-1}\Big(A_\lambda(X) \star A_\lambda(Y)\Big) = X \star Y + \lambda\, X \circ Y. \tag{1.5}$$

Equating the coefficients of λ in (1.5), we obtain the formula (0.2). It is easy to see that the transformation

$$R \longrightarrow R + ad_\star a, \tag{1.6}$$

does not change the multiplication $\circ$ for any $a \in \mathbf{V}$, where $ad_\star a$ is a linear operator $v \longrightarrow a \star v - v \star a$.

Comparing the coefficients of λ^2 in (1.5), we get the following identity

$$\begin{aligned} &R(R(X) \star Y + X \star R(Y)) - R(X) \star R(Y) - R^2(X \star Y) \\ &= S(X) \star Y + X \star S(Y) - S(X \star Y), \end{aligned} \tag{1.7}$$

for any $X, Y \in \mathbf{V}$. It is not difficult to prove that if (1.7) holds for some operators R and S then the multiplication (0.2) is associative and compatible with $\star$. Under transformation (1.6) the operator S is changing as follows

$$S \longrightarrow S + ad_\star a \circ R + \frac{1}{2}(ad_\star a)^2.$$

In the important special case $S = 0$, we have

$$R\Big(R(X) \star Y + X \star R(Y)\Big) - R(X) \star R(Y) - R^2(X \star Y) = 0. \tag{1.8}$$

In the paper [12] some examples of such R-operators have been found.

Definition. We call operators R and R' equivalent if $R - R' = ad_\star a$ for some $a \in \mathbf{V}$.

It is known that any derivative of semi-simple algebra has the form $ad_\star a$ for some $a \in \mathbf{V}$. Therefore, the formula (0.2) gives the same multiplications for operators R and R' if and only if these operators are equivalent.

Example 1.1. Let a be an arbitrary element of $\mathbf{V}$ and R be the operator of left multiplication by a with respect to $\star$. Then R satisfies (1.8) and the corresponding multiplication $X \circ Y = X \star a \star Y$ is associative and compatible with $\star$.

Example 1.2. Suppose that $\star$ is the standard matrix product in $\mathbf{V} = Mat_2$, $a, b \in \mathbf{V}$, then the product

$$X \circ Y = (aX - Xa)\,(bY - Yb)$$

is associative and compatible with the standard one. The corresponding operator R is given by $R(X) = a\,(Xb - bX)$. If $\operatorname{Det} a = 0$, then the operator R satisfies (1.8). The Example 1 from the paper [12] corresponds to the special case of the Example 1.2 where the matrices a and b are diagonal.

The following statement can be verified straightforwardly.

Proposition 1.1. The Examples 1.1 and 1.2 describe all associative multiplications compatible with the matrix product in Mat_2.

Example 1.3. Let $\mathbf{e}_1, \dots, \mathbf{e}_m$ be a basis in $\mathbf{V}$ and the multiplication $\star$ is given by

$$\mathbf{e}_i \star \mathbf{e}_j = \delta^i_j \mathbf{e}_i. \tag{1.9}$$

The algebra thus defined is commutative and semi-simple. Suppose the entries r_{ij} of the matrix R satisfy the following relations:

$$\sum_{k=1}^{m} r_{ki} = q_0, \qquad \text{and} \qquad r_{ik} r_{jk} = r_{ij} r_{jk} + r_{ji} r_{ik} \quad \text{for} \quad i \neq j \neq k \neq i,$$

where q_0 is an arbitrary constant. The generic solution of this system of algebraic equations is given by

$$r_{ii} = q_0 - \sum_{k \neq i} r_{ki}, \qquad r_{ij} = \frac{q_i p_j}{p_i - p_j}, \qquad i \neq j,$$

where p_i, q_j are arbitrary constants. The formula (0.2) defines a multiplication

$$\mathbf{e}_i \circ \mathbf{e}_j = r_{ij} \mathbf{e}_j + r_{ji} \mathbf{e}_i - \delta^i_j \sum_{k=1}^{m} r_{ik} \mathbf{e}_k$$

compatible with $\star$. Since this multiplication is linear with respect to the parameters q_i, we have got a family of $m+1$ pairwise compatible associative multiplications. This family can be described in a different way in terms of the generating function

$$\mathbf{E}(u) = \mathbf{e}_1 + u\, \mathbf{e}_2 + \ldots + u^{m-1}\, \mathbf{e}_m.$$

Let $q(u) = a_0 + u\, a_1 + \ldots + u^m\, a_m$ be an arbitrary polynomial of degree n. Define a multiplication of the generating functions by the formula

$$\mathbf{E}(u)\, \mathbf{E}(v) = \frac{u q(v)}{u - v}\, \mathbf{E}(u) + \frac{v q(u)}{v - u}\, \mathbf{E}(v). \tag{1.10}$$

It is easy to verify that (1.10) yields an associative multiplication between $\mathbf{e}_1, ..., \mathbf{e}_m$ linear with respect to the parameters $a_0, \dots, a_m$. Let $b_1, ..., b_m$ be roots of $q(u)$ and assume that these roots are pairwise distinct. Then $\tilde{\mathbf{e}}_i = b_i q'(b_i)\, \mathbf{E}(b_i)$ form a basis, in which this multiplication is given by (1.9).

The formula (1.10) admits the following generalization. Let $\mathbf{V}$ be a finite dimensional vector space with two compatible associative multiplications $\star$ and $\circ$. Let F_m be a vector space of polynomials in one variable t with degree less then m. We are going to construct $m+1$ pairwise compatible associative multiplications on the space $\mathbf{V}\otimes F_m$. For a vector $x\in\mathbf{V}$ we denote by x_i the element $x\otimes t^i\in\mathbf{V}\otimes F_m$. Denote by $x(u)$ the following polynomial in u with values in the space $\mathbf{V}\otimes F_m$:

$$x(u)=x_0+x_1u+...+x_{m-1}u^{m-1}.$$

Let us also fix an arbitrary polynomial $q(u)\in\mathbb{C}[u]$ of degree m.

Theorem 1.1. The formula

$$x(u)y(v)=\frac{q(u)}{u-v}((x\star y)(v)+v(x\circ y)(v))+\frac{q(v)}{v-u}((x\star y)(u)+u(x\circ y)(u)) \tag{1.11}$$

defines an associative multiplication on the linear space $\mathbf{V}\otimes F_m$. Here $x,y\in\mathbf{V}$ are arbitrary vectors.

Proof. It is clear that both r.h.s. and l.h.s. of (1.11) are polynomials in u,v of degree $m-1$ with values in $\mathbf{V}\otimes F_m$. Therefore, the formula defines a product in this space. Associativity of this product can be easily checked by direct calculation.

Note that the formula (1.11) defines the product which linearly depends on the polynomial $q(u)$ of degree m. Therefore, we have $m+1$ pairwise compatible associative multiplications on the space $\mathbf{V}\otimes F_m$.

Remark 1. If $\mathbf{V}=\mathbb{C}$ with trivial pair $1\star 1=0, 1\circ 1=1$, then this construction gives the Example 1.3 (see (1.10)).

Remark 2. Let $b_1,...,b_m$ be roots of $q(u)$ and assume that these roots are pairwise distinct. One can check that the algebra $\mathbf{V}\otimes F_m$ with respect to the multiplication (1.11) is isomorphic to a direct sum of m components. Moreover, the ith component is isomorphic to $\mathbf{V}$ with respect to the product $x\bullet y=x\star y+b_ix\circ y$. This is a direct consequence of the formula (1.11). In particular, if $\mathbf{V}$ is semi-simple for generic linear combination of $\star$ and $\circ$, and the roots $b_1,...,b_m$ are also generic, then $\mathbf{V}\otimes F_m$ is isomorphic to direct sum of m copies of $\mathbf{V}$.

2 Matrix case

2.1 Construction of the second product

Consider now the case where the algebra is isomorphic to Mat_n with respect to the first product. Any linear operator R on the space Mat_n may be written in the form $R(x)=a_1xb^1+...+a_lxb^l$ for some matrices $a_1,...,a_l,b^1,...,b^l$. Indeed, the operators $x\to e_{i,j}xe_{i_1,j_1}$ form a basis in the space of linear operators on Mat_n.

It is convenient to represent the operator R from the formula (0.2) in the form (0.3) with p being smallest possible in the class of equivalence of R. This means that the matrices

$\{a_1, ..., a_p, 1\}$ are linear independent as well as the matrices $\{b^1, ..., b^p, 1\}$. According to (0.2), the second product has the following form

$$x \circ y = a_i\, x\, b^i\, y + x\, a_i\, y\, b^i - a_i\, xy\, b^i + x\, c\, y. \tag{2.12}$$

Note that we have the following transformations, which do not change the class of equivalence of R. The first family of such transformations is

$$a_i \to a_i + u_i, \qquad b^i \to b^i + v^i, \qquad c \to c - u_i b^i - v^i a_i - u_i v^i$$

for any constants $u_1, ..., u_p, v^1, ..., v^p$ and the second one is

$$a_i \to g_i^k a_k, \qquad b^i \to h_k^i b^k, \qquad c \to c,$$

where $g_i^k h_k^j = \delta_i^j$. This means that we can regard a_i and b^i as bases in dual vector spaces.

Theorem 2.1. The multiplication $\circ$ given by the formula (2.12) is an associative product on the space Mat_n if and only if there exist tensors $\phi_{i,j}^k, \mu_{i,j}, \psi_k^{i,j}, \lambda^{i,j}, t_j^i$ such that the following relations hold:

$$a_i a_j = \phi_{i,j}^k a_k + \mu_{i,j}, \qquad b^i b^j = \psi_k^{i,j} b^k + \lambda^{i,j}, \tag{2.13}$$

$$b^i a_j = \psi_j^{k,i} a_k + \phi_{j,k}^i b^k + t_j^i + \delta_j^i c, \tag{2.14}$$

$$b^i c = \lambda^{k,i} a_k - t_k^i b^k - \phi_{k,l}^i \psi_s^{l,k} b^s - \phi_{k,l}^i \lambda^{l,k}, \quad c a_j = \mu_{j,k} b^k - t_j^k a_k - \phi_{k,l}^s \psi_j^{l,k} a_s - \mu_{k,l} \psi_j^{l,k} \tag{2.15}$$

Moreover, the tensors $\phi_{i,j}^k, \mu_{i,j}, \psi_k^{i,j}, \lambda^{i,j}, t_j^i$ satisfy the properties

$$\phi_{j,k}^s \phi_{s,l}^i + \mu_{j,k} \delta_l^i = \phi_{j,s}^i \phi_{k,l}^s + \delta_j^i \mu_{k,l}, \qquad \phi_{j,k}^s \mu_{i,s} = \phi_{i,j}^s \mu_{s,k}, \tag{2.16}$$

$$\psi_s^{i,j} \psi_l^{s,k} + \delta_l^k \lambda^{i,j} = \psi_s^{j,k} \psi_l^{i,s} + \delta_l^i \lambda^{j,k}, \qquad \psi_s^{i,j} \lambda^{s,k} = \psi_s^{j,k} \lambda^{i,s}, \tag{2.17}$$

$$\phi_{j,k}^s \psi_s^{l,i} = \phi_{s,k}^l \psi_j^{s,i} + \phi_{j,s}^i \psi_k^{l,s} + \delta_k^l t_j^i - \delta_j^i t_k^l - \delta_j^i \phi_{s,r}^l \psi_k^{r,s},$$

$$\phi_{j,k}^s t_s^i = \psi_j^{s,i} \mu_{s,k} + \phi_{j,s}^i t_k^s - \delta_j^i \psi_k^{s,r} \mu_{r,s}, \quad \psi_s^{k,i} t_j^s = \phi_{j,s}^i \lambda^{k,s} + \psi_j^{s,i} t_s^k - \delta_j^i \phi_{s,r}^k \lambda^{r,s} \tag{2.18}$$

Proof. Associativity $(x \circ y) \circ z = x \circ (y \circ z)$ is equivalent to the following identity

$$\begin{aligned} & a_i a_j x (b^j y b^i - y b^j b^i) z + a_i a_j x (y b^j - b^j y) z b^i + x (a_i a_j y - a_i y a_j) z b^j b^i + \\ & a_i x (y a_j - a_j y) z b^j b^i + a_i x (b^i a_j y - y b^i a_j) z b^j + a_i x (a_j y b^j b^i - b^i a_j y b^j + c y b^i - y b^i c) z + \\ & x (a_j y b^j a_i - a_i a_j y b^j - a_i y c + c a_i y) z b^i + x (a_i y b^i c - c a_i y b^i) z + a_i x (y c - c y) z b^i = 0 \end{aligned} \tag{2.19}$$

From this identity one can readily obtain (2.13), (2.14), (2.15) using the following

Lemma 2.1. Let $p_1 x q_1 + ... + p_l x q_l = 0$ for all $x \in Mat_n$. If $p_1, ..., p_l$ are linear independent matrices, then $q_1 = ... = q_l = 0$. Similarly, if $q_1, ..., q_l$ are linear independent matrices, then $p_1 = ... = p_l = 0$.

Indeed, suppose that some product $a_{i_0}a_{j_0}$ is linearly independent of $1, a_1, ..., a_p$. Since $1, a_1, ..., a_p$ are linear independent by assumption, there exists such a basis in the linear space spanned by $\{1, a_i, a_ia_j; 1 \le i, j \le p\}$ that is a subset of this set and contains the subset $\{1, a_1, ..., a_p, a_{i_0}a_{j_0}\}$. In this basis the coefficient of $a_{i_0}a_{j_0}$ has the form

$$q_{i,j}\Big((b^jyb^i - yb^jb^i)z + (yb^j - b^jy)zb^i\Big), \tag{2.20}$$

where $q_{i,j}$ are some constants not all equal to zero. Given y, z, consider the left hand side of (2.19) as a linear operator applying to the argument x. It follows from Lemma 2.1 that the coefficient (2.20) is equal to zero. Applying again Lemma 2.1 to the operator (2.20) and using the linear independence of $1, b^1, ..., b^p$, we obtain $q_{i,j} = 0$ for all i and j, which is a contradiction. Therefore, all a_ia_j are linear combinations of $1, a_1, ..., a_p$ and, similarly, all b^ib^j are linear combinations of $1, b^1, ..., b^p$. This proves (2.13). Substitute these expressions for a_ia_j and b^ib^j to (2.19) and apply Lemma 2.1 twice. Firstly, we consider the left hand side of (2.19) as a linear operator with argument x and take $1, a_1, ..., a_p$ for $p_1, ..., p_l$. After that we regard the same expression as a linear operator with argument z and take $1, b^1, ..., b^p$ for $q_1, ..., q_l$. As the result, we obtain the equation $[y,\, b^ia_j - \psi^{k,i}_j a_k - \phi^i_{j,k}b^k - \delta^i_j c] = 0$ equivalent to (2.14) and the following relations

$$\phi^k_{i,j}(b^jyb^i - yb^jb^i) + \lambda^{j,k}(ya_j - a_jy) + a_jyb^jb^i - b^ka_jyb^j + cyb^k - yb^kc = 0,$$

$$\psi^{j,i}_k(a_ia_jy - a_iya_j) + \mu_{k,j}(yb^j - b^jy) + a_jyb^ja_k - a_ka_jyb^j - a_kyc + ca_ky = 0.$$

Substituting the expressions (2.13) and (2.14) for a_ia_j, b^ib^j and b^ja_i into these relations, we get (2.15). It can be checked that all these steps are invertible and (2.19) follows from (2.13)-(2.15).

The associativity of the matrix product $a_ia_ja_k$ and the linear independence of $a_1, ..., a_p, 1$ imply (2.16). Similarly, (2.17) follows from the associativity of the product $b^ib^jb^k$ and the linear independence of $b^1, ..., b^p, 1$. The remaining identities are consequences of the associativity for $b^ia_ja_k$ and $b^ib^ja_k$.

Remark. Under conditions (2.13)-(2.15), the operator (0.3) satisfies (1.7) with

$$S(x) = \mu_{ji}\Big(b^ixb^j - \psi^{i,j}_k xb^k - \lambda^{ij}x\Big).$$

In particular, the operator (0.3) satisfies (1.8) iff $\mu_{ji} = 0$.

2.2 M-structures and corresponding associative algebras

In this subsection we describe the algebraic structure underlying Theorem 2.1.

Definition. By weak M-structure on a linear space $\mathcal{L}$ we mean the following data:

- Two subspaces $\mathcal{A}$ and $\mathcal{B}$ and distinguished element $1 \in \mathcal{A} \cap \mathcal{B} \subset \mathcal{L}$.

- A non-degenerate symmetric scalar product $(\cdot,\cdot)$ on the space $\mathcal{L}$.
- Associative products $\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ and $\mathcal{B}\times\mathcal{B}\to\mathcal{B}$ with unity 1.
- A left action $\mathcal{B}\times\mathcal{L}\to\mathcal{L}$ of the algebra $\mathcal{B}$ and a right action $\mathcal{L}\times\mathcal{A}\to\mathcal{L}$ of the algebra $\mathcal{A}$ on the space $\mathcal{L}$, which commute to each other.

These data should satisfy the following properties:

1. $\dim\mathcal{A}\cap\mathcal{B}=\dim\mathcal{L}/(\mathcal{A}+\mathcal{B})=1$. The intersection of $\mathcal{A}$ and $\mathcal{B}$ is the one dimensional space spanned by the unity 1.

2. The restriction of the action $\mathcal{B}\times\mathcal{L}\to\mathcal{L}$ to the subspace $\mathcal{B}\subset\mathcal{L}$ is the product in $\mathcal{B}$. The restriction of the action $\mathcal{L}\times\mathcal{A}\to\mathcal{L}$ to the subspace $\mathcal{A}\subset\mathcal{L}$ is the product in $\mathcal{A}$.

3. $(a_1,a_2)=(b_1,b_2)=0$ and $(b_1b_2,v)=(b_1,b_2v)$, $(v,a_1a_2)=(va_1,a_2)$ for any $a_1,a_2\in\mathcal{A}$, $b_1,b_2\in\mathcal{B}$ and $v\in\mathcal{L}$.

It follows from these properties that $(\cdot,\cdot)$ gives a non- degenerate pairing between $\mathcal{A}/\mathbb{C}1$ and $\mathcal{B}/\mathbb{C}1$, so $\dim\mathcal{A}=\dim\mathcal{B}$ and $\dim\mathcal{L}=2\dim\mathcal{A}$.

For given weak M-structure $\mathcal{L}$ we can define an algebra generated by $\mathcal{L}$ with natural compatibility and universality conditions.

Definition. By weak M-algebra associated with a weak M-structure $\mathcal{L}$ we mean an associative algebra $U(\mathcal{L})$ with the following properties:

1. $\mathcal{L}\subset\mathcal{U}(\mathcal{L})$ and the actions $\mathcal{B}\times\mathcal{L}\to\mathcal{L}$, $\mathcal{L}\times\mathcal{A}\to\mathcal{L}$ are restrictions of the product in $U(\mathcal{L})$.

2. For any algebra X with the property **1** there exist and unique a homomorphism of algebras $X\to U(\mathcal{L})$, which is identity on $\mathcal{L}$.

It is easy to see that if $U(\mathcal{L})$ exist, then it is unique for given $\mathcal{L}$. Let us describe the structure of $U(\mathcal{L})$ explicitly. Let $\{1,A_1,...,A_p\}$ be a basis of $\mathcal{A}$ and $\{1,B^1,...,B^p\}$ be a dual basis of $\mathcal{B}$ (which means that $(A_i,B^j)=\delta_i^j$). Let $C\in\mathcal{L}$ does not belong to the sum of $\mathcal{A}$ and $\mathcal{B}$. Since $(\cdot,\cdot)$ is non- degenerate, we have $(1,C)\neq 0$. Multiplying C by constant, we can assume that $(1,C)=1$. Adding linear combination of $1,A_1,...,A_p,B^1,...,B^p$ to C, we can assume that $(C,C)=(C,A_i)=(C,B^j)=0$. Such element C is uniquely determined by choosing basis in $\mathcal{A}$ and $\mathcal{B}$.

Lemma 2.2. The algebra $U(\mathcal{L})$ is defined by the following relations

$$A_iA_j=\phi^k_{i,j}A_k+\mu_{i,j},\qquad B^iB^j=\psi^{i,j}_kB^k+\lambda^{i,j}\tag{2.21}$$

$$B^iA_j=\psi^{k,i}_jA_k+\phi^i_{j,k}B^k+t^i_j+\delta^i_jC,\tag{2.22}$$

$$B^iC=\lambda^{k,i}A_k+u^i_kB^k+p^i,\qquad CA_j=\mu_{j,k}B^k+u^k_jA_k+q_i\tag{2.23}$$

for certain tensors $\phi^k_{i,j},\psi^{i,j}_k,\mu_{i,j},\lambda^{i,j},u^i_k,p^i,q_i$.

Proof. Relations (2.21) just mean that $\mathcal{A}$ and $\mathcal{B}$ are associative algebras. Since $\mathcal{L}$ is a left $\mathcal{B}$-module and a right $\mathcal{A}$-module, the products B^iA_j, CA_j, B^iC should be linear combinations of the basis elements $1, A_1, ..., A_p, B^1, ..., B^p, C$. Applying property **3** of weak M-structure, we obtain required form of these products. The universality condition of $U(\mathcal{L})$ shows that this algebra is defined by (2.21) - (2.23).

Let us define an element $K \in U(\mathcal{L})$ by the formula $K = A_iB^i + C$. It is clear that K thus defined does not depend on the choice of the basis in $\mathcal{A}$ and $\mathcal{B}$ provided $(A_i, B^j) = \delta_i^j$, $(1, C) = 1$ and $(C, C) = (C, A_i) = (C, B^j) = 0$. Indeed, the coefficients of K are just entries of the tensor inverse to the form $(\cdot, \cdot)$.

Definition. Weak M-structure $\mathcal{L}$ is called M-structure if $K \in U(\mathcal{L})$ is a central element of the algebra $U(\mathcal{L})$.

Lemma 2.3. For any M-structure $\mathcal{L}$ we have

$$p^i = -\phi^i_{k,l}\lambda^{l,k}, \qquad q_i = -\psi_i^{k,l}\mu_{l,k}, \qquad u_i^j = -t_i^j - \phi^j_{k,l}\psi_i^{l,k}.$$

Proof. This is a direct consequence of the identities $A_iK = KA_i$ and $B^jK = KB^j$.

Lemma 2.4. For M-structure $\mathcal{L}$ the algebra $U(\mathcal{L})$ is defined by the generators $A_1, ..., A_p$, $B^1, ..., B^p$ and relations obtained from (2.21), (2.22) by elimination of C. Tensors $\phi^k_{i,j}$, $\psi_k^{i,j}$, $\mu_{i,j}$, $\lambda^{i,j}$ should satisfy the properties (2.16), (2.17), (2.18). Any algebra defined by such generators and relations is isomorphic to $U(\mathcal{L})$ for a suitable M-structure $\mathcal{L}$.

Theorem 2.2. Let $\mathcal{L}$ be an M-structure. Then for any representation $U(\mathcal{L}) \to Mat_n$ given by $A_1 \to a_1, ..., A_p \to a_p, B^1 \to b^1, ..., B^p \to b^p, C \to c$ the formula (2.12) defines an associative product on Mat_n compatible with the usual product.

Proof. Comparing (2.13)-(2.15) with (2.21)-(2.23), where p^i, q_i and u_i^j are given by Lemma 2.3, we see that this is just reformulation of the Theorem 2.1.

Definition. A representation of $U(\mathcal{L})$ is called non-degenerate if the matrices $a_1, ..., a_p, 1$ are linear independent as well as $b^1, ..., b^p, 1$.

Remark. It is clear that M-structure $\mathcal{L}'$ is equivalent to $\mathcal{L}$ if the defining relations for $U(\mathcal{L}')$ can be obtained from the defining relations for $U(\mathcal{L})$ by a transformation of the form

$$A_i \to g_i^kA_k + u_i, \quad B^i \to h_k^iB^k + v^i, \quad C \to C - u_ih_k^iB^k - v^ig_i^kA_k - u_iv^i$$

where $u_1, ..., u_p, v^1, ..., v^p$ are some constants and $g_i^kh_k^j = \delta_i^j$.

Theorem 2.3. There is an one-to-one correspondence between n dimensional non-degenerate representations of algebras $U(\mathcal{L})$ corresponding to M-structures up to equivalence of M-structures and associative products on Mat_n compatible with the usual product.

Proof. This is a direct consequence of Theorems 2.1 and 2.2.

The structure of the algebra $U(\mathcal{L})$ for M-structure $\mathcal{L}$ is described by the following

Theorem 2.4. The algebra $U(\mathcal{L})$ is spanned by the elements K^s, A_iK^s, B_jK^s, $A_iB^jK^s$, where $i, j = 1, ..., p,$ and $s = 0, 1, 2, ...$

Proof. Since K is a central element, we have $KA_i = A_iK,\ KB^j = B^jK,\ KC = CK$. Using this, one can check that a product of any elements listed in the theorem can be written as a linear combination of these elements. To prove the theorem one should also check associativity, which is possible to do directly.

Remark. As we have mentioned in the Introduction, if a linear space $\mathbf{V}$ is equipped with two compatible associative multiplications, then one can construct those in the space $Mat_m(\mathbf{V})$. Since $Mat_m(Mat_n) = Mat_{mn}$, in the matrix case this construction yields a second multiplication for the algebras $Mat_{mn},\ m = 1, 2, ...$ if we have a second multiplication in Mat_n. One can see that in the language of representations of M-structures this corresponds to the direct sum of m copies of a given n-dimensional representation.

Example 2.1. Suppose $\mathcal{A}$ and $\mathcal{B}$ are generated by elements $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $A^{p+1} = B^{p+1} = 1$. Take $1, A, ..., A^p, B, ..., B^p, C$ for a basis in $\mathcal{L}$ and assume that $(B^i, A^{-i}) = \epsilon^i - 1$, $(1, C) = 1$ and other scalar products are equal to zero. Here ϵ is a primitive root of unity of order $p+1$. The structures of left $\mathcal{B}$-module and right $\mathcal{A}$-module on $\mathcal{L}$ are defined by the formulas:

$$B^iA^j = \frac{\epsilon^{-j} - 1}{\epsilon^{-i-j} - 1}A^{i+j} + \frac{\epsilon^i - 1}{\epsilon^{i+j} - 1}B^{i+j},$$

for $i + j \neq 0$ modulo p and

$$B^iA^{-i} = 1 + (\epsilon^i - 1)C,$$

$$CA^i = \frac{1}{1 - \epsilon^i}A^i + \frac{1}{\epsilon^i - 1}B^i,$$

$$B^iC = \frac{1}{\epsilon^{-i} - 1}A^i + \frac{1}{1 - \epsilon^{-i}}B^i$$

for $i \neq 0$ modulo $p+1$. One can see that these formulas define an M-structure. The central element has the following form $K = C + \sum_{0<i<p} \frac{1}{\epsilon^i - 1}A^{-i}B^i$.

Let a, t be linear operators in some vector space. Assume that $a^{p+1} = 1$, $at = \epsilon ta$ and the operator $t - 1$ is invertible. It is easy to check that the formulas

$$A \to a, \qquad B \to \frac{\epsilon t - 1}{t - 1}a, \qquad C \to \frac{t}{t - 1}$$

define a representation of the algebra $U(\mathcal{L})$. Note that we do not assume that $t^{p+1} = 1$. We have only $at^{p+1} = t^{p+1}a$ which easily follows from the commutation relation between a and t.

2.3 Case of commutative semi-simple algebra $\mathcal{A}$

Consider the case

$$A_iA_j = \delta_{i,j}A_i. \tag{2.24}$$

In other words

$$\phi^k_{i,j} = \delta_{i,j}\delta_{i,k}, \qquad \mu_{i,j} = 0. \tag{2.25}$$

Theorem 2.5. In this case any corresponding algebra $\mathcal{B}$ can be reduced by an appropriate shift $B^i \to B^i + c_i$ to one defined by the formulas:

$$B^i B^j = (u_i - q_{i,j})B^i + q_{i,j}B^j + v_i, \qquad i \neq j \tag{2.26}$$

and

$$(B^i)^2 = u_i B^i + v_i, \tag{2.27}$$

where constants $u_i, v_i, q_{i,j}$ satisfy the following relations:

$$q^2_{i,j} = u_i q_{i,j} + v_i, \tag{2.28}$$

$$(u_i - q_{i,j})^2 = u_j(u_i - q_{i,j}) + v_j, \tag{2.29}$$

where $i \neq j$, and

$$(q_{i,k} - q_{j,k})(q_{i,k} - q_{i,j}) = 0 \tag{2.30}$$

for pairwise distinct i, j, k. The corresponding algebra $U(\mathcal{L})$ is determined by the formulas (2.24), (2.26), (2.27) and:

$$B^i A_j = (u_j - q_{j,i})A_j \qquad \text{for} \quad i \neq j, \qquad\qquad B^i A_i = u_i A_i + \sum_{k\neq i} q_{k,i}A_k + B^i + C,$$

$$B^i C = \sum_{1\leq k\leq p} v_k A_k - u_i B^i - v_i, \qquad CA_j = -u_j A_j.$$

Proof. In our case the first equation of (2.18) reads $\delta_{j,k}t^i_k = \delta_{i,j}t^i_k$, which gives $t^i_j = \delta^i_j r_j$ for some tensor r_j. The third equation of (2.17) reads $\delta_{j,k}\psi^{l,i}_j = \delta_{l,k}\psi^{l,i}_j + \delta_{i,j}\psi^{l,i}_k + (r_j - r_k)\delta^l_k\delta^i_j - \delta^i_j\psi^{l,l}_k$, which has the following general solution $\psi^{l,i}_j = \delta^l_j(h_j - r_i - q_{l,i}) + \delta^i_j q_{l,i}$. From the second equation of (2.18) we find $\lambda^{k,j} = \lambda^{k,k} + q_{k,j}(r_j - r_k)$. Substituting these into the formulas for the product in the algebra $\mathcal{B}$, we get (2.26) and (2.27) after suitable shift of $B^1, ..., B^p$ for some u_i, v_i. Associativity of the algebra $\mathcal{B}$ gives (2.28), (2.29) and (2.30). Indeed, consider an algebra defined by identities

$$B_i B_j = p_{ij}B_i + q_{ij}B_j + r_{ij}, \quad i \neq j, \qquad\qquad B_i^2 = u_i B_i + v_i$$

This algebra is associative iff

$$r_{ij} = -p_{ij}q_{ij}, \qquad q^2_{ij} = u_i q_{ij} + v_i,$$

$$(p_{ij} - q_{jk})(p_{ik} - p_{jk}) = 0, \qquad (p_{ij} - q_{jk})(q_{ik} - q_{ij}) = 0,$$

which equivalent to (2.28), (2.29) and (2.30) in our case. The explicit form of identities for the algebra $U(\mathcal{L})$ follows from (2.22) and (2.23).

Remark. It follows from (2.26), (2.27) that the vector space spanned by 1 and B^i, where i belongs to arbitrary subset of the set $\{1, 2, \ldots, p\}$, is a subalgebra in $\mathcal{B}$.

Two algebras (2.26) - (2.30) are said to be equivalent if they are related by a transformation of the form

$$B^i \to c_1 B^i + c_2, \qquad i = 1, \ldots, p \tag{2.31}$$

and a permutation of the generators $B^1, \ldots, B^p$.

Example 2.2. Suppose $\mathcal{B}$ is commutative. It follows from

$$B^i B^j - B^j B^i = (u_i - q_{i,j} - q_{j,i})B^i - (u_j - q_{i,j} - q_{j,i})B^j + (v_i - v_j) \tag{2.32}$$

that in this case we have $u_i = u_j$, $v_i = v_j$ and $q_{i,j} + q_{j,i} = u_i$ for any i, j. Such an algebra is equivalent to the one defined by

$$u_1 = \cdots = u_p = 0, \qquad v_1 = \cdots = v_p = 1, \qquad q_{ij} = 1, \quad q_{ji} = -1, \quad i > j.$$

It is easy to verify that this algebra is semi-simple.

Example 2.3. One solution of the system (2.28) - (2.30) is obvious:

$$q_{ij} = u_i + \tau, \qquad v_i = \tau^2 + u_i \tau, \qquad i, j = 1, \ldots, p,$$

where τ is arbitrary parameter. Using transformation (2.31), we can reduce τ by zero. Algebra $\mathcal{B}$ described in this example is called *regular*. The corresponding associative product compatible with the matrix product in Mat_{p+1} have been independently found by I.Z. Golubchik.

Now our aim is to describe all irregular algebras $\mathcal{B}$. It follows from (2.28), (2.29) that

$$(q_{ki} - q_{kj})(q_{ki} + q_{kj} - u_k) = 0 \tag{2.33}$$

for any distinct i, j, k and

$$(q_{ij} - u_i)(u_i - u_j) = v_i - v_j \tag{2.34}$$

for any $i \neq j$. Formula (2.34) implies that

$$(q_{ij} - q_{ji} - u_i + u_j)(u_i - u_j) = 0. \tag{2.35}$$

We associate with any algebra $\mathcal{B}$ the following equivalence relation on the set $\{1, 2, \ldots, p\}$: $i \sim j$ if $u_i = u_j$. Denote by m the number of equivalence classes. It follows from (2.34) that if i and j belong to the same equivalence class then $v_i = v_j$. Furthermore, if i and j belong to different equivalence classes, we have

$$q_{ij} = u_i + \frac{v_i - v_j}{u_i - u_j} \tag{2.36}$$

and therefore q_{ij} is well defined function on the set of pairs of equivalence classes.

Besides $\sim$, we consider one more relation $\approx$ defined as follows: $i \approx j$ if $i = j$ or if $u_i = u_j$ and $q_{ij} = q_{ji}$. It is easy to derive from (2.30) that $\approx$ is an equivalence relation and the value of q_{ij} does not depend on the choosing of i, j from the equivalence class.

Case m=1. Consider the case $m = 1$ or, the same $u_i = u_1$ for any i. Denote one of two possible values of q_{ij} by $-\tau$. It follows from (2.28) that other possible value is equal to $u_1 + \tau$ and $v_1 = \tau^2 + u_1\tau$. If $\mathcal{B}$ is irregular then $-\tau$ and $u_1 + \tau$ are distinct.

Given u_1, τ any algebra $\mathcal{B}$ is defined by the following data: arbitrary clustering of the set $\{1, 2, \ldots, p\}$ into equivalence classes $K_1, \ldots, K_s$ with respect to $\approx$ and arbitrary function Q_{ij} on the pairs of equivalence classes with values in $\{-\tau, u_1 + \tau\}$ such that $Q_{\alpha\beta} \neq Q_{\beta\alpha}$ if $\alpha \neq \beta$. The function q_{ij} is defined as follows: $q_{i,j} = Q_{ij}$ if i, j belong to different classes and $q_{i,j} = Q_{\alpha\alpha}$ if $i, j \in K_\alpha$. It can be verified that the parameters defined as above satisfy (2.28) - (2.30).

Case m=2. In this case we have two distinct parameters u_1 and u_2. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be corresponding equivalence classes with respect to $\sim$. Denote $\frac{v_2 - v_1}{u_2 - u_1}$ by τ. Then $v_i = \tau^2 + u_i\tau$. Using (2.36), we get $q_{ik} = u_\alpha + \tau$ if $i \in \mathcal{K}_\alpha$ and $k \notin \mathcal{K}_\alpha$. It follows from (2.30) that for any j, q_{ij} may take on values $u_\alpha + \tau$ or $-\tau$. Suppose $i, j \in \mathcal{K}_\alpha$ and $k \notin \mathcal{K}_\alpha$; then (2.30) yields

$$(q_{ij} - \tau - u_1)(q_{ij} - \tau - u_2) = 0.$$

Therefore if $\tau + u_1 \neq -\tau$ and $\tau + u_2 \neq -\tau$, then $\mathcal{B}$ is regular. In the case $\tau = -\frac{u_2}{2}$, we have $q_{ij} = \frac{u_2}{2}$ for any $i \in \mathcal{K}_2$. Formula (2.36) implies $q_{ji} = u_1 - \frac{u_2}{2}$. To complete the description we should define q_{ij} for $i, j \in \mathcal{K}_1$. It is clear that the vector space spanned by 1 and B^i, where $i \in \mathcal{K}_1$, is a subalgebra that belongs to the case $m = 1$ described above. It turns out that this subalgebra can be chosen arbitrarily.

Case m $\geq$ 3. In this case all possible algebras $\mathcal{B}$ can be described as follows.

Proposition 2.1. Suppose $u_1, \ldots, u_m$ are pairwise distinct and $m \geq 3$. Then if $p > 3$ then $\mathcal{B}$ is regular. For $p = 3$ there exists one more algebra described in the following

Example 2.4. The algebra $\mathcal{B}$ defined by relations

$$\begin{gathered} q_{21} = q_{31}, \qquad q_{12} = q_{32}, \qquad q_{13} = q_{23}, \\ u_1 = q_{12} + q_{13}, \qquad u_2 = q_{21} + q_{23}, \qquad u_3 = q_{31} + q_{32}, \\ v_1 = -q_{12}q_{13}, \qquad v_2 = -q_{21}q_{23}, \qquad v_3 = -q_{31}q_{32} \end{gathered} \tag{2.37}$$

is isomorphic to Mat_2.

Proof. Suppose that u_i, u_j, u_k are pairwise distinct. Then we deduce from (2.35) that

$$q_{ij} - q_{ji} - u_i + u_j = q_{jk} - q_{kj} - u_j + u_k = q_{ki} - q_{ik} - u_k + u_i = 0$$

and therefore $q_{ij} + q_{jk} + q_{ki} = q_{ji} + q_{ik} + q_{kj}$. It follows from this formula and (2.30) that there exist only two possibilities:

$$q_{ji} = q_{ki}, \qquad q_{ij} = q_{kj}, \qquad q_{ik} = q_{jk} \tag{2.38}$$

or

$$q_{ij} = q_{ik}, \qquad q_{jk} = q_{ji}, \qquad q_{ki} = q_{kj}. \tag{2.39}$$

It is not difficult to show that $\mathcal{B}$ is regular if it contains a triple of the type (2.39). It can be verified also that if $\mathcal{B}$ contains a triple of the type (2.38), then $\mathcal{B}$ coincides with the algebra described in Example 2.4.

Remark. It follows from Theorem 4.2 of Section 4 that if $\mathcal{A}$ is commutative and semi-simple and $\mathcal{B}$ is semi-simple, then either $\mathcal{B}$ is commutative (Example 2.2) or $\mathcal{B} = Mat_2$ (Example 2.4).

3 Semi-simple case

Consider an associative algebra $M = \oplus_{1\le\alpha\le m} M_\alpha$, where M_α is isomorphic to Mat_{n_α}. We are going to study associative products in this algebra compatible with the usual one. All constructions and results are similar to the matrix case.

We use Greek letters for indexes related to the direct summands of M. Throughout this section, we keep the following summation agreement. We sum by repeated Latin indexes and do not sum by repeated Greek indexes if the opposite is not stated explicitly. Symbols δ_i^j, $\delta_{i,j}$, and $\delta^{i,j}$ stand for the Kronecker delta.

3.1 Construction of the second product

Let R be a linear operator in the space M. The operator R takes $x_\alpha \in M_\alpha$ to $\sum_\beta R_{\beta,\alpha}(x_\alpha)$, where $R_{\beta,\alpha}(x_\alpha) \in M_\beta$. It is clear that $R_{\beta,\alpha}$ is a linear map from M_α to M_β. Note that any linear map from the space Mat_{n_α} to the space Mat_{n_β} can be written in the form $x_\alpha \to a_{i,\beta,\alpha} x_\alpha b^i_{\alpha,\beta}$, where $a_{1,\beta,\alpha}, ..., a_{l,\beta,\alpha}$ are some $n_\beta \times n_\alpha$ matrices and $b^1_{\alpha,\beta}, ..., b^l_{\alpha,\beta}$ are some $n_\alpha \times n_\beta$ matrices.

Assume that $R_{\beta,\alpha}(x_\alpha) = a_{i,\beta,\alpha} x_\alpha b^i_{\alpha,\beta}$ for $\alpha \neq \beta$ and $R_{\alpha,\alpha}(x_\alpha) = a_{i,\alpha,\alpha} x_\alpha b^i_{\alpha,\alpha} + c_\alpha x_\alpha$ for some matrices $a_{i,\beta,\alpha}, b^i_{\alpha,\beta}, 1 \le i \le p_{\alpha,\beta}$ and c_α, $1 \le \alpha, \beta \le m$ with $p_{\alpha,\beta}$ being smallest possible in the equivalence class of R. This means that the following sets of matrices are linear independent: $\{a_{1,\beta,\alpha}, ..., a_{p_{\alpha,\beta},\beta,\alpha}\}$, $\{b^1_{\alpha,\beta}, ..., b^{p_{\alpha,\beta}}_{\alpha,\beta}\}$ for $\alpha \neq \beta$ and $\{1, a_{1,\alpha,\alpha}, ..., a_{p_{\alpha,\alpha},\alpha,\alpha}\}$, $\{1, b^1_{\alpha,\alpha}, ..., b^{p_{\alpha,\alpha}}_{\alpha,\alpha}\}$. It follows from (0.2) that the second product of $x_\alpha \in M_\alpha$ and $y_\beta \in M_\beta$ has the form

$$\begin{aligned} x_\alpha \circ y_\beta &= a_{i,\beta,\alpha} x_\alpha b^i_{\alpha,\beta} y_\beta + x_\alpha a_{i,\alpha,\beta} y_\beta b^i_{\beta,\alpha}, \qquad \alpha \neq \beta, \\ x_\alpha \circ y_\alpha &= a_{i,\alpha,\alpha} x_\alpha b^i_{\alpha,\alpha} y_\alpha + x_\alpha a_{i,\alpha,\alpha} y_\alpha b^i_{\alpha,\alpha} - a_{i,\alpha,\alpha} x_\alpha y_\alpha b^i_{\alpha,\alpha} + x_\alpha c_\alpha y_\alpha. \end{aligned} \tag{3.40}$$

We have the following transformations preserving the equivalence class of R. The first family of such transformations is defined by

$$a_{i,\alpha,\alpha} \to a_{i,\alpha,\alpha} + u_{i,\alpha}, \qquad b^i_{\alpha,\alpha} \to b^i_{\alpha,\alpha} + v^i_\alpha, \qquad c_\alpha \to c_\alpha - u_{i,\alpha} b^i_{\alpha,\alpha} - v^i_\alpha a_{i,\alpha,\alpha} - u_{i,\alpha} v^i_\alpha$$

for any constants $u_{1,\alpha}, ..., u_{p_{\alpha,\alpha},\alpha}, v_\alpha^1, ..., v_\alpha^{p_{\alpha,\alpha}}$ and the second one is given by

$$a_{i,\beta,\alpha} \to g^k_{i,\alpha,\beta} a_{k,\beta,\alpha}, \qquad b^i_{\alpha,\beta} \to h^i_{k,\alpha,\beta} b^k_{\alpha,\beta}, \qquad c_\alpha \to c_\alpha,$$

where $g^k_{i,\alpha,\beta} h^j_{k,\alpha,\beta} = \delta^j_i$. This means that we can regard $a_{i,\beta,\alpha}$ and $b^i_{\alpha,\beta}$ as bases in dual vector spaces.

Theorem 3.1. If $\circ$ is an associative product on the space M, then

$$a_{i,\alpha,\beta} a_{j,\beta,\gamma} = \phi^k_{i,j,\alpha,\beta,\gamma} a_{k,\alpha,\gamma} + \delta_{\alpha,\gamma} \mu_{i,j,\alpha,\beta}, \qquad b^i_{\alpha,\beta} b^j_{\beta,\gamma} = \psi^{i,j}_{k,\alpha,\beta,\gamma} b^k_{\alpha,\gamma} + \delta_{\alpha,\gamma} \lambda^{i,j}_{\alpha,\beta}, \tag{3.41}$$

$$b^i_{\alpha,\beta} a_{j,\beta,\gamma} = \phi^i_{j,k,\beta,\gamma,\alpha} b^k_{\alpha,\gamma} + \psi^{k,i}_{j,\gamma,\alpha,\beta} a_{k,\alpha,\gamma} + \delta_{\alpha,\gamma} t^i_{j,\alpha,\beta} + \delta_{\alpha,\gamma} \delta^i_j c_\alpha, \tag{3.42}$$

$$c_\alpha a_{i,\alpha,\beta} = \mu_{i,k,\alpha,\beta} b^k_{\alpha,\beta} - t^k_{i,\beta,\alpha} a_{k,\alpha,\beta} - \sum_{1\le\nu\le m} \phi^k_{l,s,\alpha,\nu,\beta} \psi^{s,l}_{i,\beta,\nu,\alpha} a_{k,\alpha,\beta} - \sum_{1\le\nu\le m} \delta_{\alpha,\beta} \mu_{l,s,\alpha,\nu} \psi^{s,l}_{i,\alpha,\nu,\alpha}, \tag{3.43}$$

$$b^i_{\alpha,\beta} c_\beta = \lambda^{k,i}_{\beta,\alpha} a_{k,\alpha,\beta} - t^i_{k,\alpha,\beta} b^k_{\alpha,\beta} - \sum_{1\le\nu\le m} \phi^i_{l,s,\beta,\nu,\alpha} \psi^{s,l}_{k,\alpha,\nu,\beta} b^k_{\alpha,\beta} - \sum_{1\le\nu\le m} \delta_{\alpha,\beta} \phi^i_{s,l,\alpha,\nu,\alpha} \lambda^{l,s}_{\alpha,\nu,\alpha}, \tag{3.44}$$

where $\phi^k_{i,j,\alpha,\beta,\gamma}, \mu_{i,j,\alpha,\beta}, \psi^{i,j}_{k,\alpha,\beta,\gamma}, \lambda^{i,j}_{\alpha,\beta}, t^i_{j,\alpha,\beta}$ are tensors satisfying the properties

$$\phi^s_{j,k,\alpha,\beta,\gamma} \phi^i_{s,l,\alpha,\gamma,\delta} + \delta^i_l \delta_{\alpha,\gamma} \mu_{j,k,\alpha,\beta} = \phi^i_{j,s,\alpha,\beta,\delta} \phi^s_{k,l\beta,\gamma,\delta} + \delta^i_j \delta_{\beta,\delta} \mu_{k,l,\beta,\gamma}, \tag{3.45}$$

$$\phi^s_{j,k\alpha,\beta,\gamma} \mu_{i,s,\alpha,\gamma} = \phi^s_{i,j,\beta,\gamma,\alpha} \mu_{s,k,\alpha,\beta},$$

$$\psi^{i,j}_{s,\alpha,\beta,\gamma} \psi^{s,k}_{l,\alpha,\gamma,\delta} + \delta^k_l \delta_{\alpha,\gamma} \lambda^{i,j}_{\alpha,\beta} = \psi^{j,k}_{s,\beta,\gamma,\delta} \psi^{i,s}_{l,\alpha,\beta,\delta} + \delta^i_l \delta_{\beta,\delta} \lambda^{j,k}_{\beta,\gamma}, \tag{3.46}$$

$$\psi^{i,j}_{s,\alpha,\beta,\gamma} \lambda^{s,k}_{\alpha,\gamma} = \psi^{j,k}_{s,\beta,\gamma,\alpha} \lambda^{i,s}_{\alpha,\beta},$$

$$\phi^s_{j,k,\beta,\gamma,\delta} \psi^{l,i}_{s,\delta,\alpha,\beta} = \phi^l_{s,k,\alpha,\gamma,\delta} \psi^{s,i}_{j,\gamma,\alpha,\beta} + \phi^i_{j,s,\beta,\gamma,\alpha} \psi^{l,s}_{k,\delta,\alpha,\gamma} +$$
$$\delta^l_k \delta_{\alpha,\gamma} t^i_{j,\alpha,\beta} - \delta^i_j \delta_{\alpha,\gamma} t^l_{k,\delta,\alpha} - \delta^i_j \delta_{\alpha,\gamma} \textstyle\sum_{1\le\nu\le m} \phi^l_{s,r,\alpha,\nu,\delta} \psi^{r,s}_{k,\delta,\nu,\alpha},$$

$$\phi^s_{j,k,\beta,\gamma,\alpha} t^i_{s,\alpha,\beta} = \psi^{s,i}_{j,\gamma,\alpha,\beta} \mu_{s,k,\alpha,\gamma} + \phi^i_{j,s,\beta,\gamma,\alpha} t^s_{k,\alpha,\gamma} - \delta^i_j \delta_{\alpha,\gamma} \sum_{1\le\nu\le m} \psi^{s,r}_{k,\alpha,\nu,\alpha} \mu_{r,s,\alpha,\nu}, \tag{3.47}$$

$$\psi^{k,i}_{s,\alpha,\beta,\gamma} t^s_{j,\alpha,\gamma} = \phi^i_{j,s,\gamma,\alpha,\beta} \lambda^{k,s}_{\alpha,\beta} + \psi^{s,i}_{j,\alpha,\beta,\gamma} t^k_{s,\alpha,\beta} - \delta^i_j \delta_{\alpha,\beta} \sum_{1\le\nu\le m} \phi^k_{s,r,\alpha,\nu,\alpha} \lambda^{r,s}_{\alpha,\nu}$$

Proof is similar to the matrix case. Instead of Lemma 2.1 one can use the following

Lemma 3.1. Let $x \to p_1 x q_1 + ... + p_l x q_l$ be a zero map from Mat_α to Mat_β. If $p_1, ..., p_l$ are linear independent matrices, then $q_1 = ... = q_l = 0$. Similarly, if $q_1, ..., q_l$ are linear independent matrices, then $p_1 = ... = p_l = 0$.

3.2 PM-structures and corresponding associative algebras

In this subsection we describe the algebraic structure underlying Theorem 3.1.

Definition. By weak PM-structure (of size m) on a linear space $\mathcal{L}$ we mean the following data.

- Two subspaces $\mathcal{A}$ and $\mathcal{B}$ and a distinguished element $1 \in \mathcal{A} \cap \mathcal{B} \subset \mathcal{L}$.
- A non-degenerate symmetric scalar product $(\cdot, \cdot)$ on the space $\mathcal{L}$.
- Associative products $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$ with unity 1.
- A left action $\mathcal{B} \times \mathcal{L} \to \mathcal{L}$ of the algebra $\mathcal{B}$ and a right action $\mathcal{L} \times \mathcal{A} \to \mathcal{L}$ of the algebra $\mathcal{A}$ on the space $\mathcal{L}$, which commute with each other.

These data should satisfy the following properties:

1. $\dim \mathcal{A} \cap \mathcal{B} = \dim \mathcal{L}/(\mathcal{A} + \mathcal{B}) = m$. The intersection of $\mathcal{A}$ and $\mathcal{B}$ is a m-dimensional algebra isomorphic to $\mathbb{C} \oplus ... \oplus \mathbb{C}$.

2. The restriction of the action $\mathcal{B} \times \mathcal{L} \to \mathcal{L}$ to the subspace $\mathcal{B} \subset \mathcal{L}$ is the product in $\mathcal{B}$. The restriction of the action $\mathcal{L} \times \mathcal{A} \to \mathcal{L}$ to the subspace $\mathcal{A} \subset \mathcal{L}$ is the product in $\mathcal{A}$.

3. $(a_1, a_2) = (b_1, b_2) = 0$ and $(b_1 b_2, v) = (b_1, b_2 v)$, $(v, a_1 a_2) = (v a_1, a_2)$ for any $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and $v \in \mathcal{L}$.

It follows from these properties that $(\cdot, \cdot)$ defines a non- degenerate pairing between $\mathcal{A}/\mathcal{A} \cap \mathcal{B}$ and $\mathcal{B}/\mathcal{A} \cap \mathcal{B}$, so $\dim \mathcal{A} = \dim \mathcal{B}$ and $\dim \mathcal{L} = 2 \dim \mathcal{A}$.

Lemma 3.2. Let $\{e_\alpha; 1 \le \alpha \le m\}$ be a basis of the space $\mathcal{A} \cap \mathcal{B}$ such that

$$e_\alpha e_\beta = \delta_{\alpha,\beta} e_\alpha. \tag{3.48}$$

Denote by $\mathcal{L}_{\alpha,\beta}$ the vector space consisting of elements $v_{\alpha,\beta} \in \mathcal{L}$ with the property

$$e_\alpha v_{\alpha,\beta} = v_{\alpha,\beta} e_\beta = v_{\alpha,\beta}. \tag{3.49}$$

Let $\mathcal{A}_{\alpha,\beta} = \mathcal{A} \cap \mathcal{L}_{\alpha,\beta}$ and $\mathcal{B}_{\alpha,\beta} = \mathcal{B} \cap \mathcal{L}_{\alpha,\beta}$. Then the following properties hold:

- $\mathcal{L} = \oplus_{1 \le \alpha,\beta \le m} \mathcal{L}_{\alpha,\beta}$, $\quad \mathcal{A} = \oplus_{1 \le \alpha,\beta \le m} \mathcal{A}_{\alpha,\beta}$ and $\mathcal{B} = \oplus_{1 \le \alpha,\beta \le m} \mathcal{B}_{\alpha,\beta}$.
- $\dim \mathcal{A}_{\alpha,\beta} \cap \mathcal{B}_{\alpha,\beta} = \dim \mathcal{L}/(\mathcal{A}_{\alpha,\beta} + \mathcal{B}_{\alpha,\beta}) = \delta_{\alpha,\beta}$. The intersection of $\mathcal{A}_{\alpha,\alpha}$ and $\mathcal{B}_{\alpha,\alpha}$ is an one-dimensional space spanned by e_α.
- $\mathcal{B}_{\alpha,\beta} \mathcal{L}_{\beta',\gamma} = 0$ for $\beta \ne \beta'$ and $\mathcal{B}_{\alpha,\beta} \mathcal{L}_{\beta,\gamma} \subset \mathcal{L}_{\alpha,\gamma}$. Similarly $\mathcal{L}_{\alpha,\beta} \mathcal{A}_{\beta',\gamma} = 0$ for $\beta \ne \beta'$ and $\mathcal{L}_{\alpha,\beta} \mathcal{A}_{\beta,\gamma} \subset \mathcal{L}_{\alpha,\gamma}$. In particular, $\mathcal{A}_{\alpha,\beta} \mathcal{A}_{\beta',\gamma} = \mathcal{B}_{\alpha,\beta} \mathcal{B}_{\beta',\gamma} = 0$ for $\beta \ne \beta'$ and $\mathcal{A}_{\alpha,\beta} \mathcal{A}_{\beta,\gamma} \subset \mathcal{A}_{\alpha,\gamma}$, $\mathcal{B}_{\alpha,\beta} \mathcal{B}_{\beta,\gamma} \subset \mathcal{B}_{\alpha,\gamma}$.

- $\mathcal{L}_{\alpha,\beta} \perp \mathcal{L}_{\beta',\alpha'}$ if $\alpha \neq \alpha'$ or $\beta \neq \beta'$.

It follows from these properties that $(\cdot,\cdot)$ gives non- degenerate pairing between $\mathcal{A}_{\alpha,\beta}$ and $\mathcal{B}_{\beta,\alpha}$ for $\alpha \neq \beta$ and between $\mathcal{A}_{\alpha,\alpha}/\mathbb{C}e_\alpha$ and $\mathcal{B}_{\alpha,\alpha}/\mathbb{C}e_\alpha$. Therefore $\dim \mathcal{A}_{\alpha,\beta} = \dim \mathcal{B}_{\beta,\alpha}$.

Proof. It is clear that $1 = e_1 + ... + e_m$. For $v \in \mathcal{L}$ we have $v = 1v1 = \sum_{1\le\alpha,\beta\le m} e_\alpha v e_\beta$ and $e_\alpha v e_\beta \in \mathcal{L}_{\alpha,\beta}$, which proves all statements of Lemma 3.2.

Definition. By weak PM-algebra associated with a weak PM-structure $\mathcal{L}$ we mean an associative algebra $U(\mathcal{L})$ possessing the following properties:

1. $\mathcal{L} \subset \mathcal{U}(\mathcal{L})$ and the actions $\mathcal{B} \times \mathcal{L} \to \mathcal{L}$, $\mathcal{L} \times \mathcal{A} \to \mathcal{L}$ are the restrictions of the product in $U(\mathcal{L})$.

2. For any algebra X with the property **1** there exists a unique homomorphism of algebras $X \to U(\mathcal{L})$ identical on $\mathcal{L}$.

It is easy to see that if $U(\mathcal{L})$ exists, then it is unique for given $\mathcal{L}$. Let us describe the structure of $U(\mathcal{L})$ explicitly. Let $\{e_\alpha, A_{i,\alpha,\alpha}; 1 \le i \le p_{\alpha,\alpha}\}$ be a basis of $\mathcal{A}_{\alpha,\alpha}$ and $\{e_\alpha, B^i_{\alpha,\alpha}; 1 \le i \le p_{\alpha,\alpha}\}$ be the dual basis of $\mathcal{B}_{\alpha,\alpha}$. Let $\{A_{i,\alpha,\beta}; 1 \le i \le p_{\beta,\alpha}\}$ be a basis of $A_{\alpha,\beta}$ for $\alpha \neq \beta$ and $\{B^i_{\beta,\alpha}; 1 \le i \le p_{\beta,\alpha}\}$ be the dual basis of $B_{\beta,\alpha}$. This means that $(A_{i,\alpha,\beta}, B^j_{\beta',\alpha'}) = \delta^j_i \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}$. Take $C_\alpha \in \mathcal{L}_{\alpha,\alpha}$ that does not belong to the sum of $\mathcal{A}_{\alpha,\alpha}$ and $\mathcal{B}_{\alpha,\alpha}$. Since $(\cdot,\cdot)$ is non-degenerate, we have $(e_\alpha, C_\alpha) \neq 0$. Multiplying C_α by constant, we can assume that $(e_\alpha, C_\alpha) = 1$. Adding a linear combination of $e_\alpha, A_{1,\alpha,\alpha}, ..., A_{p_{\alpha,\alpha},\alpha,\alpha}, B^1_{\alpha,\alpha}, ..., B^{p_{\alpha,\alpha}}_{\alpha,\alpha}$ to C_α, we can assume that $(C_\alpha, C_\alpha) = (C_\alpha, A_{i,\alpha,\alpha}) = (C_\alpha, B^j_{\alpha,\alpha}) = 0$. Such element C_α is uniquely determined by choosing of basis in $\mathcal{A}_{\alpha,\alpha}$.

Lemma 3.3. The algebra $U(\mathcal{L})$ is defined by (3.48), (3.49) and the following relations

$$A_{i,\alpha,\beta} A_{j,\beta,\gamma} = \phi^k_{i,j,\alpha,\beta,\gamma} A_{k,\alpha,\gamma} + \delta_{\alpha,\gamma} \mu_{i,j,\alpha,\beta}, \qquad B^i_{\alpha,\beta} B^j_{\beta,\gamma} = \psi^{i,j}_{k,\alpha,\beta,\gamma} B^k_{\alpha,\gamma} + \delta_{\alpha,\gamma} \lambda^{i,j}_{\alpha,\beta} \tag{3.50}$$

$$B^i_{\alpha,\beta} A_{j,\beta,\gamma} = \phi^i_{j,k,\beta,\gamma,\alpha} B^k_{\alpha,\gamma} + \psi^{k,i}_{j,\gamma,\alpha,\beta} A_{k,\alpha,\gamma} + \delta_{\alpha,\gamma} t^i_{j,\alpha,\beta} + \delta_{\alpha,\gamma} \delta^i_j C_\alpha \tag{3.51}$$

$$C_\alpha A_{i,\alpha,\beta} = \mu_{i,k,\alpha,\beta} B^k_{\alpha,\beta} + u^k_{i,\beta,\alpha} A_{k,\alpha,\beta} + \delta_{\alpha,\beta} p^i_\alpha \tag{3.52}$$

$$B^i_{\alpha,\beta} C_\beta = \lambda^{k,i}_{\beta,\alpha} A_{k,\alpha,\beta} + u^i_{k,\alpha,\beta} B^k_{\alpha,\beta} + \delta_{\alpha,\beta} q_{i,\alpha} \tag{3.53}$$

for certain tensors $\phi^k_{i,j,\alpha,\beta,\gamma}, \mu_{i,j,\alpha,\beta}, \psi^{i,j}_{k,\alpha,\beta,\gamma}, \lambda^{i,j}_{\alpha,\beta}, t^i_{j,\alpha,\beta}, u^k_{i,\beta,\alpha}, p^i_\alpha, q_{i,\alpha}$.

Proof. Relations (3.50) just mean that $\mathcal{A}$ and $\mathcal{B}$ are associative algebras. Since $\mathcal{L}$ is a left $\mathcal{B}$-module and a right $\mathcal{A}$-module, $B^i_{\alpha,\beta} A_{j,\beta,\gamma}, C_\alpha A_{j,\alpha,\beta}, B^i_{\alpha,\beta} C_\beta$ should be linear combinations of the basis elements in $\mathcal{L}$. Applying properties **1, 2, 3** of weak PM-structure and Lemma 3.2, we obtain required form of these products. The universality condition of $U(\mathcal{L})$ shows that this algebra is defined by (3.48), (3.49), (3.50) - (3.53).

It is clear that $U(\mathcal{L}) = \oplus_{1\le\alpha,\beta\le m} U(\mathcal{L})_{\alpha,\beta}$, where $U(\mathcal{L})_{\alpha,\beta} = \{v \in U(\mathcal{L}); e_\alpha v = v e_\beta = v\}$. We have $U(\mathcal{L})_{\alpha,\beta} U(\mathcal{L})_{\beta',\gamma} = 0$ for $\beta \ne \beta'$ and $U(\mathcal{L})_{\alpha,\beta} U(\mathcal{L})_{\beta,\gamma} \subset U(\mathcal{L})_{\alpha,\gamma}$.

Let us define an element $K_\alpha \in U(\mathcal{L})$ by the formula $K_\alpha = C_\alpha + \sum_{1\le\nu\le m} A_{i,\alpha,\nu} B^i_{\nu,\alpha}$. It is clear that K_α thus defined does not depend on the choice of the basis in $\mathcal{A}$ and $\mathcal{B}$ provided $(A_{i,\alpha,\beta}, B^j_{\beta',\alpha'}) = \delta^j_i \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}$, $(e_\alpha, C_\alpha) = 1$ and $(C_\alpha, C_\alpha) = (C_\alpha, A_{i,\alpha,\alpha}) = (C_\alpha, B^j_{\alpha,\alpha}) = 0$. Indeed, the coefficients of K_α are just entries of the tensor inverse to the form $(\cdot,\cdot)$.

Definition. A weak PM-structure $\mathcal{L}$ is called PM-structure if $K = \sum_{1\le\alpha\le m} K_\alpha \in U(\mathcal{L})$ is a central element of the algebra $U(\mathcal{L})$.

It is clear that K is central if and only if $K_\alpha v = v K_\beta$ for all $v \in U(\mathcal{L})_{\alpha,\beta}$.

Lemma 3.4. For any PM-structure $\mathcal{L}$, we have

$$p^i_\alpha = -\sum_{1\le\nu\le m} \phi^i_{s,l,\alpha,\nu,\alpha} \lambda^{l,s}_{\alpha,\nu,\alpha}, \qquad q_{i,\alpha} = -\sum_{1\le\nu\le m} \mu_{l,s,\alpha,\nu} \psi^{s,l}_{i,\alpha,\nu,\alpha},$$

$$u^j_{i,\alpha,\beta} = -t^j_{i,\alpha,\beta} - \sum_{1\le\nu\le m} \phi^j_{l,s,\beta,\nu,\alpha} \psi^{s,l}_{i,\alpha,\nu,\beta}$$

Proof. This is a direct consequence of $A_{i,\alpha,\beta} K_\beta = K_\alpha A_{i,\alpha,\beta}$ and $B^j_{\alpha,\beta} K_\beta = K_\alpha B^j_{\alpha,\beta}$.

Lemma 3.5. For any PM-structure $\mathcal{L}$, the algebra $U(\mathcal{L})$ is defined by the generators $\{e_\alpha, A_{i,\alpha,\beta}, B^i_{\beta,\alpha}; 1 \le i \le p_{\beta,\alpha}, 1 \le \alpha,\beta \le m\}$ and relations obtained from (3.48), (3.50), (3.51) by elimination of C_α. Tensors $\phi^k_{i,j}, \psi^{i,j}_k, \mu_{i,j}, \lambda^{i,j}$ should satisfy the properties (3.45)-(3.47). Any algebra defined by such generators and relations is isomorphic to $U(\mathcal{L})$ for a suitable PM-structure $\mathcal{L}$.

Let $\tau : U(\mathcal{L}) \to End(V)$ be a representation of the algebra $U(\mathcal{L})$. Let $\pi_\alpha = \tau(e_\alpha)$ and $V_\alpha = \pi_\alpha(V)$. It follows from (3.48) that $V = \oplus_{1\le\alpha\le m} V_\alpha$. Let $n_\alpha = dim(V_\alpha)$. We can regard $x \in U(\mathcal{L})_{\alpha,\beta}$ as a linear operator from V_β to V_α or, choosing basis in $V_1, ..., V_m$, as an $n_\alpha \times n_\beta$ matrix.

Definition. By a representation of a PM-algebra $U(\mathcal{L})$ of dimension $(n_1, ..., n_m)$ we mean a correspondence $A_{i,\beta,\alpha} \to a_{i,\beta,\alpha}$, $B^i_{\alpha,\beta} \to b^i_{\alpha,\beta}$, $C_\alpha \to c_\alpha$; $1 \le i \le p_{\alpha,\beta}, 1 \le \alpha,\beta \le m$, where $a_{i,\alpha,\beta}$, $b^i_{\alpha,\beta}$ are $n_\alpha \times n_\beta$ matrices and c_α are $n_\alpha \times n_\alpha$ matrices satisfying (3.41), (3.42), (3.43), (3.44).

It is clear that this definition is equivalent to the usual one for the associative algebra $U(\mathcal{L})$.

Theorem 3.2. Let $\mathcal{L}$ be a PM-structure. Then for any representation of $U(\mathcal{L})$ given by $A_{i,\beta,\alpha} \to a_{i,\beta,\alpha}, B^i_{\alpha,\beta} \to b^i_{\alpha,\beta}, C_\alpha \to c_\alpha;\ 1 \le i \le p_{\alpha,\beta}, 1 \le \alpha,\beta \le m$ the formula (3.40) defines an associative product on $M = \oplus_{1\le\alpha\le m} Mat_{n_\alpha}$ compatible with the usual one.

Proof. Comparing (3.41)-(3.44) with (3.50)-(3.53), where $p^i_\alpha, q_{i,\alpha}$ and $u^j_{i,\alpha,\beta}$ are given by Lemma 3.4, we see that this is just reformulation of Theorem 3.1.

Definition. A representation of a PM-algebra $U(\mathcal{L})$ is called non-degenerate if the following sets of matrices are linear independent: $\{1, a_{i,\alpha,\alpha}; 1 \le i \le p_{\alpha,\alpha}\}$, $\{1, b_{i,\alpha,\alpha}; 1 \le i \le p_{\alpha,\alpha}\}$,

$\{a_{i,\beta,\alpha}; 1 \le i \le p_{\alpha,\beta}\}$ and $\{b^i_{\alpha,\beta}; 1 \le i \le p_{\alpha,\beta}\}$ for $\alpha \ne \beta$.

Theorem 3.3. There is a one-to-one correspondence between $(n_1, ..., n_m)$-dimensional non-degenerate representations of PM-algebras $U(\mathcal{L})$ up to equivalence of the algebras and associative products on $Mat_{n_1} \oplus ... \oplus Mat_{n_m}$ compatible with the usual product.

Proof. This is a direct consequence of Theorems 3.1 and 3.2.

The structure of a PM-algebra $U(\mathcal{L})$ can be described as follows.

Theorem 3.4. A basis of $U(\mathcal{L})_{\alpha,\beta}$ for $\alpha \ne \beta$ consists of the elements

$$\{A_{i,\alpha,\beta}K^s_\beta,\ \ B^j_{\alpha,\beta}K^s_\beta,\ \ A_{i_1,\alpha,\nu}B^{j_1}_{\nu,\beta}K^s_\beta\},$$

where $1 \le i \le p_{\beta,\alpha},\ 1 \le j \le p_{\alpha,\beta},\ 1 \le \alpha, \beta, \nu \le m, 1 \le i_1 \le p_{\nu,\alpha},\ 1 \le j_1 \le p_{\nu,\beta},\ s = 0, 1, 2, \ldots$. A basis of $U(\mathcal{L})_{\alpha,\alpha}$ consists of the elements

$$\{e_\alpha,\ \ A_{i,\alpha,\alpha}K^s_\alpha,\ \ B^j_{\alpha,\alpha}K^s_\alpha,\ \ A_{i_1,\alpha,\nu}B^{j_1}_{\nu,\alpha}K^s_\alpha\},$$

where $1 \le i, j \le p_{\alpha,\alpha},\ 1 \le \nu \le m,\ 1 \le i_1, j_1 \le p_{\nu,\alpha},\ s = 0, 1, 2, \ldots$.

Proof. Since K is a central element, we have $K_\alpha A_{i,\alpha,\beta} = A_{i,\alpha,\beta}K_\beta$, $\quad K_\alpha B^j_{\alpha,\beta} = B^j_{\alpha,\beta}K_\beta$, $K_\alpha C_\alpha = C_\alpha K_\alpha$. Using this, one can check that a product of any elements listed in the theorem can be written as a linear combination of these elements. To prove the theorem, one should also check the associativity, which is possible to do directly.

Definition. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be weak PM-structures. Let $\mathcal{A}_1, \mathcal{B}_1 \subset \mathcal{L}_1$ and $\mathcal{A}_2, \mathcal{B}_2 \subset \mathcal{L}_2$ be corresponding algebras and $(\cdot,\cdot)_1$, $(\cdot,\cdot)_2$ be corresponding scalar products. By direct sum of $\mathcal{L}_1$ and $\mathcal{L}_2$ we mean the weak PM-structure $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$ with $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$ and $(\cdot,\cdot) = (\cdot,\cdot)_1 + (\cdot,\cdot)_2$. We assume the componentwise action of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{L}$.

Definition. A weak PM-structure is called indecomposable if it is not equal to $\mathcal{L}_1 \oplus \mathcal{L}_2$ for nonzero $\mathcal{L}_1$ and $\mathcal{L}_2$.

It is clear that decomposable PM-structures correspond to decomposable pairs of compatible associative products.

Definition. Let $\mathcal{L}$ be a weak PM-structure. By the opposite weak PM-structure $\mathcal{L}^{op}$ we mean a PM-structure with the same linear space $\mathcal{L}$, the same scalar product and algebras $\mathcal{A}$, $\mathcal{B}$ replaced by the opposite algebras $\mathcal{B}^{op}$, $\mathcal{A}^{op}$, correspondingly. We remind that a right module over an associative algebra is left module over opposite algebra and vice-versa.

Let us describe the PM-structure related to Example 1.3.

Example 3.1. Let $\dim \mathcal{A}_{\alpha,\beta} = \dim \mathcal{B}_{\alpha,\beta} = 1$ for all $1 \le \alpha, \beta \le m$. Suppose that for $\alpha \ne \beta$ the space $\mathcal{A}_{\alpha,\beta}$ is spanned by an element $A_{\alpha,\beta}$ and the space $\mathcal{B}_{\alpha,\beta}$ is spanned by an element $B_{\alpha,\beta}$. Note that $\mathcal{A}_{\alpha,\alpha} = \mathcal{B}_{\alpha,\alpha} = \mathbb{C}e_\alpha$. Assume that $A_{\alpha,\beta}A_{\beta,\gamma} = A_{\alpha,\gamma}$, $B_{\alpha,\beta}B_{\beta,\gamma} = B_{\alpha,\gamma}$ for $\alpha \ne \gamma$ and $A_{\alpha,\beta}A_{\beta,\alpha} = B_{\alpha,\beta}B_{\beta,\alpha} = e_\alpha$. Note that $\dim \mathcal{L}_{\alpha,\beta} = 2$ for all $1 \le \alpha, \beta \le m$ and a basis of $\mathcal{L}_{\alpha,\beta}$ is $\{A_{\alpha,\beta}, B_{\alpha,\beta}\}$ for $\alpha \ne \beta$. A basis of $\mathcal{L}_{\alpha,\alpha}$ is $\{e_\alpha, C_\alpha\}$. Assume that $(A_{\alpha,\beta}, B_{\beta,\alpha}) = (u_\alpha - u_\beta)/t_\beta$,

$(e_\alpha, C_\alpha) = t_\alpha^{-1}$ and structures of left $\mathcal{B}$-module and right $\mathcal{A}$-module are given by the formulas:

$$B_{\alpha,\beta}A_{\beta,\gamma} = \frac{u_\beta - u_\gamma}{u_\alpha - u_\gamma}A_{\alpha,\gamma} + \frac{u_\beta - u_\alpha}{u_\gamma - u_\alpha}B_{\alpha,\gamma}$$

for $\alpha \neq \gamma$ and

$$B_{\alpha,\beta}A_{\beta,\alpha} = e_\alpha + (u_\beta - u_\alpha)C_\alpha,$$

$$C_\alpha A_{\alpha,\beta} = \frac{1}{u_\alpha - u_\beta}A_{\alpha,\beta} + \frac{1}{u_\beta - u_\alpha}B_{\alpha,\beta}, \qquad B_{\alpha,\beta}C_\beta = \frac{1}{u_\alpha - u_\beta}A_{\alpha,\beta} + \frac{1}{u_\beta - u_\alpha}B_{\alpha,\beta}.$$

One can check that these formulas determine a PM-structure on the space $\mathcal{L}$ for generic $u_1, ..., u_m$, $t_1, ..., t_m$. The elements

$$K_\alpha = t_\alpha C_\alpha + \sum_{\beta\neq\alpha} \frac{t_\beta}{u_\alpha - u_\beta}(A_{\alpha,\beta}B_{\beta,\alpha} - e_\alpha)$$

satisfy the property $K_\alpha v = vK_\beta$ for all $v \in U(\mathcal{L})_{\alpha,\beta}$.

Note that the corresponding algebra $U(\mathcal{L})$ has one-dimensional representation $A_{\alpha,\beta} \to 1$, $B_{\alpha,\beta} \to u_\beta/u_\alpha$, $C_\alpha \to 1/u_\alpha$, which gives rise to Example 1.3.

4 Case of semi-simple algebras $\mathcal{A}$ and $\mathcal{B}$

4.1 Matrix of multiplicities

In this Subsection we suppose that $\mathcal{L}$ is a weak PM-structure. We use a notation V^l for a direct sum of l copies of a linear space V if $l \in \mathbb{N}$ and assume $V^0 = 0$. We recall that any left $End(V)$-module has the form V^l and any right $End(V)$-module has the form $(V^\star)^l$.

Lemma 4.1. Let $\mathcal{A}$ be a semi-simple algebra, namely $\mathcal{A} = \oplus_{1\le i\le r}End(V_i)$, where $\dim V_i = m_i$. Then $\mathcal{L}$ as $\mathcal{A}$-module is isomorphic to $\oplus_{1\le i\le r}(V_i^\star)^{2m_i}$.

Proof. It is known that any right $\mathcal{A}$-module has the form $\oplus_{1\le i\le r}(V_i^\star)^{l_i}$ for some $l_1, ..., l_r \ge 0$. Therefore $\mathcal{L} = \oplus_{1\le i\le r}\mathcal{L}_i$ where $\mathcal{L}_i = (V_i^\star)^{l_i}$. Note that $\mathcal{A} \subset \mathcal{L}$ and, moreover, $End(V_i) \subset \mathcal{L}_i$ for $i = 1, ..., r$. Besides, $End(V_i) \perp \mathcal{L}_j$ for $i \neq j$. Indeed, we have $(v, a) = (v, Id_i a) = (vId_i, a) = 0$ for $v \in \mathcal{L}_j$ and $a \in End(V_i)$, where Id_i is the unity of the subalgebra $End(V_i)$. Since $(\cdot, \cdot)$ is non-degenerate and $End(V_i) \perp End(V_i)$ by the property 3 of weak PM-structure, we have $\dim \mathcal{L}_i \ge 2\dim End(V_i)$. But $\sum_i \dim \mathcal{L}_i = \dim \mathcal{L} = 2\dim \mathcal{A} = \sum_i 2\dim End(V_i)$ and we obtain the identity $\dim \mathcal{L}_i = 2\dim End(V_i)$ for each $i = 1, ..., r$, which is equivalent to the statement of Lemma 4.1.

Lemma 4.2. Let $\mathcal{A}$ and $\mathcal{B}$ be semi-simple, namely

$$\mathcal{A} = \oplus_{1\le i\le r}End(V_i), \qquad \mathcal{B} = \oplus_{1\le j\le s}End(W_j), \qquad \dim V_i = m_i, \quad \dim W_j = n_j.$$

Then $\mathcal{L}$ as $\mathcal{A}\otimes\mathcal{B}$-module is isomorphic to $\oplus_{1\le i\le r,1\le j\le s}(V_i^\star\otimes W_j)^{a_{i,j}}$, where $a_{i,j}\ge 0$ and

$$\sum_j a_{i,j}n_j=2m_i,\qquad \sum_i a_{i,j}m_i=2n_j. \tag{4.54}$$

Proof. It is known that any $\mathcal{A}\otimes\mathcal{B}$-module has the form $\oplus_{1\le i\le r,1\le j\le s}(V_i^\star\otimes W_j)^{a_{i,j}}$, where $a_{i,j}\ge 0$. Applying Lemma 4.1, we obtain $\dim\mathcal{L}_i=2m_i^2$, where $\mathcal{L}_i=\oplus_{1\le j\le s}(V_i^\star\otimes W_j)^{a_{i,j}}$. This gives the first equation from (4.54). The second equation can be obtained similarly.

Definition. The matrix $(a_{i,j})$ from Lemma 4.2 is called matrix of multiplicities of a weak PM-structure $\mathcal{L}$.

Definition. An $r\times s$ matrix $(a_{i,j})$ is called decomposable if there exist partitions $\{1,...,r\}=I\sqcup I'$ and $\{1,...,s\}=J\sqcup J'$ such that $a_{i,j}=0$ for $(i,j)\in I\times J'\sqcup I'\times J$.

Lemma 4.3. If matrix of multiplicities is decomposable, then corresponding PM-structure is decomposable.

Proof. Suppose $(a_{i,j})$ is decomposable. We have $\mathcal{A}=\mathcal{A}'\oplus\mathcal{A}''$, $\mathcal{B}=\mathcal{B}'\oplus\mathcal{B}''$ and $\mathcal{L}=\mathcal{L}'\oplus\mathcal{L}''$ where

$$\mathcal{A}'=\oplus_{i\in I}End(V_i),\qquad \mathcal{A}''=\oplus_{i\in I'}End(V_i),\qquad \mathcal{B}'=\oplus_{j\in J}End(W_j),$$
$$\mathcal{B}''=\oplus_{j\in J'}End(W_j),\qquad \mathcal{L}'=\oplus_{(i,j)\in I\times J}(V_i^\star\otimes W_j)^{a_{i,j}},\qquad \mathcal{L}''=\oplus_{(i,j)\in I'\times J'}(V_i^\star\otimes W_j)^{a_{i,j}}.$$

It is clear that this is a decomposition of $\mathcal{L}$.

Definition. We call an $r\times s$ matrix with non-negative integral entries $(a_{i,j})$ admissible if it is indecomposable and (4.54) holds for some positive vectors $(m_1,...,m_r)$ and $(n_1,...,n_s)$.

Now our aim is to classify all admissible matrices. Note that if A is admissible, then A^t is also admissible. Moreover, if A is the matrix of multiplicities of a weak PM structure with semi-simple algebras $\mathcal{A}$ and $\mathcal{B}$, then A^t is the matrix of multiplicities of the opposite weak PM-structure.

Theorem 4.1. There is a one-to-one correspondence between the following two sets:

1. Admissible matrices up to a permutation of rows and columns.

2. Simple laced affine Dynkin diagrams with a partition of the set of vertices into two subsets (represented by black and white circles in the pictures below) such that vertices in each subset are pairwise non-connected.

Namely, assign to each vertex of such a Dynkin diagram a vector space from the set $\{V_1,...,V_r,W_1,...,W_s\}$ in such a way that there is a one-to-one correspondence between this set and the set of vertices, and for any i,j the spaces V_i, V_j are not connected by edges as well as the spaces W_i, W_j. Then $a_{i,j}$ is equal to the number of edges between V_i and W_j.

Proof. Let $(a_{i,j})$ be an admissible $r\times s$ matrix. Consider a linear space with a basis $\{v_1,...,v_r,w_1,...,w_s\}$ and the symmetric bilinear form $(v_i,v_j)=(w_i,w_j)=2\delta_{i,j}$, $(v_i,w_j)=-a_{i,j}$. Let $J=m_1v_1+...+m_rv_r+n_1w_1+...+n_sw_s$. It is clear that the equations (4.54) can be

written as follows $(v_i, J) = (w_j, J) = 0$, which means that J belongs to the kernel of the form $(\cdot, \cdot)$. Therefore (see [10]), the matrix of the form is a Cartan matrix of a simple laced affine Dynkin diagram.

On the other hand, consider a simple laced affine Dynkin diagram with a partition of the set of vertices into two subsets such that vertices of the same subset are not connected. It is clear that if such a partition exists, then it is unique up to transposition of subsets. Let $v_1, ..., v_r$ be roots corresponding to vertices of the first subset and $w_1, ..., w_s$ be roots corresponding to the second subset. We have $(v_i, v_j) = (w_i, w_j) = 2\delta_{i,j}$. Let $a_{i,j} = -(v_i, w_j)$ and $J = m_1 v_1 + ... + m_r v_r + n_1 w_1 + ... + n_s w_s$ be an imaginary root. It is clear that (4.54) holds and therefore $(a_{i,j})$ is admissible.

Note that the transposition of the subsets corresponds to the transposition of matrix $(a_{i,j})$.

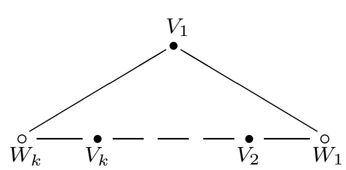
Applying known classification of affine Dynkin diagrams [11], we obtain the following

Theorem 4.2. Let $A = (a_{i,j})$ be an $r \times s$ matrix of multiplicities for a weak PM-structure. Then, after a possible permutation of rows and columns and the transposition, a matrix A is equal to one in the following list:

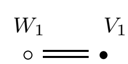
1. $A = (2)$. Here $r = s = 1$, $n_1 = m_1 = m$. The corresponding Dynkin diagram is of the type $\tilde{A}_1$.

2. $a_{i,i} = a_{i,i+1} = 1$ and $a_{i,j} = 0$ for other pairs i, j. Here $r = s = k \geq 2$, the indexes are taken modulo k, and $n_i = m_i = m$. The corresponding Dynkin diagram is $\tilde{A}_{2k-1}$.

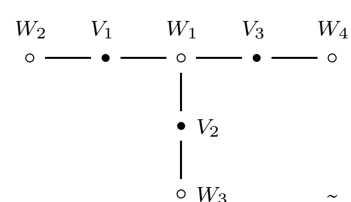
3. $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. Here $r = 3$, $s = 4$ and $n_1 = 3m$, $n_2 = n_3 = n_4 = m$, $m_1 = m_2 = m_3 = 2m$. The Dynkin diagram is $\tilde{E}_6$:



$\tilde{A}_{2k-1}$



$\tilde{A}_1$



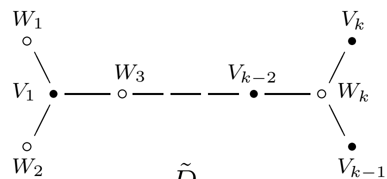
$\tilde{E}_6$

4. $A = (1, 1, 1, 1)$. Here $r = 1, s = 4$ and $n_1 = n_2 = n_3 = n_4 = m$, $m_1 = 2m$. The corresponding Dynkin diagram is $\tilde{D}_4$.

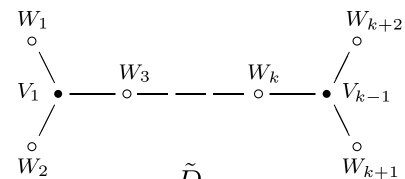
5. $a_{1,1} = a_{1,2} = a_{1,3} = 1$, $a_{2,3} = a_{2,4} = a_{3,4} = a_{3,5} = \cdots = a_{k-2,k-1} = a_{k-2,k} = 1$, $a_{k-1,k} = a_{k-1,k+1} = a_{k-1,k+2} = 1$, and $a_{i,j} = 0$ for other (i, j). Here we have $r = k - 1$, $s = k + 2$ and $n_1 = n_2 = n_{k+1} = n_{k+2} = m$, $n_3 = \cdots = n_k = 2m$, $m_1 = \cdots = m_k = 2m$. The

corresponding Dynkin diagram is $\tilde{D}_{2k}$, where $k \geq 3$.

6. $a_{1,1} = a_{1,2} = a_{1,3} = 1, \ a_{2,3} = a_{2,4} = a_{3,4} = a_{3,5} = \cdots = a_{k-2,k-1} = a_{k-2,k} = 1,$ $a_{k-1,k} = a_{k,k} = 1$, and $a_{i,j} = 0$ for other (i, j). Here we have $r = s = k \geq 3$, $n_1 = n_2 = m$, $n_3 = \cdots = n_k = 2m$, $m_1 = \cdots = m_{k-2} = 2m$, $m_{k-1} = m_k = m$. The corresponding Dynkin diagram is $\tilde{D}_{2k-1}$. Note that if $k = 3$, then $a_{1,1} = a_{1,2} = a_{1,3} = 1$, $a_{2,3} = a_{3,3} = 1$.



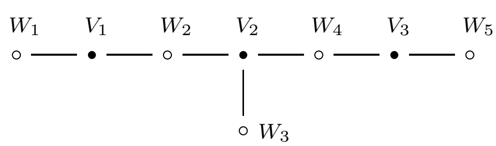
$\tilde{D}_{2k-1}$



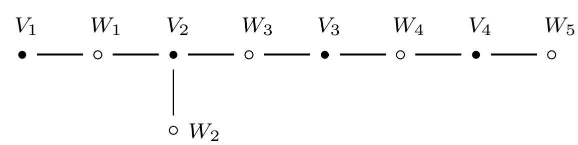
$\tilde{D}_{2k}$

7. $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$. Here $r = 3, s = 5$ and $n_1 = m$, $n_2 = 3m$, $n_3 = 2m$, $n_4 = 3m$, $n_5 = m$, $m_1 = 2m$, $m_2 = 4m$, $m_3 = 2m$. The Dynkin diagram is $\tilde{E}_7$.

8. $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$. Here $r = 4, s = 5$ and $n_1 = 4m$, $n_2 = 3m$, $n_3 = 5m$, $n_4 = 3m$, $n_5 = m$, $m_1 = 2m$, $m_2 = 6m$, $m_3 = 4m$, $m_4 = 2m$. The Dynkin diagram is $\tilde{E}_8$.



$\tilde{E}_7$



$\tilde{E}_8$

4.2 PM-structures connected with affine Dynkin diagrams

In the previous Subsection, we have shown that if $\mathcal{L}$ is an indecomposable PM-structure with semi-simple algebras $\mathcal{A} = \oplus_{1 \leq i \leq r} End(V_i)$, $\mathcal{B} = \oplus_{1 \leq j \leq s} End(W_j)$, then there exists an affine Dynkin diagram of the type A, D, or E such that:

1. There is a one-to-one correspondence between the set of vertices and the set of vector spaces $\{V_1, ..., V_r, W_1, ..., W_s\}$.

2. For any i, j the spaces V_i, V_j are not connected by edges as well as W_i, W_j.

3. $\mathcal{L}$ as $\mathcal{A}\otimes\mathcal{B}$-module is isomorphic to $\oplus_{1\le i\le r, 1\le j\le s}(V_i^\star\otimes W_j)^{a_{i,j}}$, where $a_{i,j}$ is equal to the number of edges between V_i and W_j.

4. The vector $(\dim V_1, ..., \dim V_r, \dim W_1, ..., \dim W_s)$ is an imaginary positive root of the Dynkin diagram.

To describe the corresponding PM-structure it remains to construct an embedding $\mathcal{A}\to\mathcal{L}$, $\mathcal{B}\to\mathcal{L}$ and a scalar product $(\cdot,\cdot)$ on the space $\mathcal{L}$. Note that if we fix an element $1\in\mathcal{L}$, then we can define the embedding $\mathcal{A}\to\mathcal{L}$, $\mathcal{B}\to\mathcal{L}$ by the formula $a\to 1a$, $b\to b1$ for $a\in\mathcal{A}$, $b\in\mathcal{B}$. After that it is not difficult to construct a scalar product. Moreover, we may assume that 1 is a generic element of $\mathcal{L}$. Therefore, to study PM-structures corresponding to a Dynkin diagram, one should take a generic element in $\mathcal{L}=\oplus_{1\le i\le r,1\le j\le s}(V_i^\star\otimes W_j)^{a_{i,j}}$, find its simplest canonical form by choosing bases in the vector spaces $V_1, ..., V_r, W_1, ..., W_s$, calculate the embedding $\mathcal{A}\to\mathcal{L}$, $\mathcal{B}\to\mathcal{L}$ and the scalar product $(\cdot,\cdot)$ on the space $\mathcal{L}$.

For example, consider the case $\tilde{A}_{2k-1}$. We have $\dim V_i=\dim W_i=m$ for $1\le i\le k$. Let $\{v_{i,\alpha}; 1\le\alpha\le m\}$ be a basis of $V_i^\star$ and $\{w_{i,\alpha}; 1\le\alpha\le m\}$ be a basis of W_i. Let $\{e_{i,\alpha,\beta}; 1\le\alpha,\beta\le m\}$ be a basis of $End(V_i)$ such that $v_{i,\alpha}e_{i,\alpha',\beta}=\delta_{\alpha,\alpha'}v_{i,\beta}$ and $\{f_{i,\alpha,\beta}; 1\le\alpha,\beta\le m\}$ be a basis of $End(W_i)$ such that $f_{i,\alpha,\beta}w_{i,\beta'}=\delta_{\beta,\beta'}w_{i,\alpha}$. A generic element $1\in\mathcal{L}$ in a suitable basis in V_i, W_i can be written in the form $1=\sum_{1\le i\le k,1\le\alpha\le m}(v_{i,\alpha}\otimes w_{i,\alpha}+\lambda_\alpha v_{i+1,\alpha}\otimes w_{i,\alpha})$, where index i is taken modulo k and $\lambda_1, ..., \lambda_m\in\mathbb{C}$ are generic complex numbers. The embedding $\mathcal{A}\to\mathcal{L}$, $\mathcal{B}\to\mathcal{L}$ is the following: $e_{i,\alpha,\beta}\to 1e_{i,\alpha,\beta}=v_{i,\beta}\otimes w_{i,\alpha}+\lambda_\alpha v_{i,\beta}\otimes w_{i-1,\alpha}$, $f_{i,\alpha,\beta}\to f_{i,\alpha,\beta}1=v_{i,\beta}\otimes w_{i,\alpha}+\lambda_\beta v_{i+1,\beta}\otimes w_{i,\alpha}$. It is clear that $\dim\mathcal{A}\cap\mathcal{B}=m$ and a basis of this space is $\{\sum_i(v_{i,\alpha}\otimes w_{i,\alpha}+\lambda_\alpha v_{i,\alpha}\otimes w_{i-1,\alpha}); 1\le\alpha\le m\}$. It is also clear that the algebra $\mathcal{A}\cap\mathcal{B}$ is isomorphic to $\mathbb{C}^m$. Let us introduce a new basis in the algebras $\mathcal{A}$ and $\mathcal{B}$. Namely, let $A^i_{\alpha,\beta}=\sum_{1\le j\le k}\epsilon^{ij}e_{i,\alpha,\beta}$ and $B^i_{\alpha,\beta}=\sum_{1\le j\le k}\epsilon^{ij}f_{i,\alpha,\beta}$. Here $\epsilon=\exp(2\pi i/k)$ is a primitive root of unity of degree k. Simple calculations give now the following description of the corresponding PM-structure in the case $\tilde{A}_{2k-1}$.

The algebra $\mathcal{A}$ has a basis $\{A^i_{\alpha,\beta}; 1\le\alpha,\beta\le m, i\in\mathbb{Z}/k\mathbb{Z}\}$ such that $A^i_{\alpha,\beta}A^j_{\beta,\gamma}=A^{i+j}_{\alpha,\gamma}$. The algebra $\mathcal{B}$ has a basis $\{B^i_{\alpha,\beta}; 1\le\alpha,\beta\le m, i\in\mathbb{Z}/k\mathbb{Z}\}$ such that $B^i_{\alpha,\beta}B^j_{\beta,\gamma}=B^{i+j}_{\alpha,\gamma}$. The intersection $\mathcal{A}\cap\mathcal{B}$ has a basis $\{e_\alpha=A^0_{\alpha,\alpha}=B^0_{\alpha,\alpha}; 1\le\alpha\le m\}$. A basis of the space $\mathcal{L}$ consists of the elements e_α, $A^i_{\alpha,\beta}$, $B^i_{\alpha,\beta}$, where $i\neq 0$ if $\alpha=\beta$ and C_α, where $1\le\alpha\le m$. The scalar product has the form $(B^i_{\alpha,\beta}, A^{-i}_{\beta,\alpha})=(\epsilon^i\lambda_\alpha-\lambda_\beta)/t_\alpha$, $(e_\alpha, C_\alpha)=t_\alpha^{-1}$. The action of $\mathcal{A}$ and $\mathcal{B}$ on the space $\mathcal{L}$ is given by the formulas:

$$B^i_{\alpha,\beta}A^j_{\beta,\gamma}=\frac{\epsilon^{-j}\lambda_\gamma-\lambda_\beta}{\epsilon^{-i-j}\lambda_\gamma-\lambda_\alpha}A^{i+j}_{\alpha,\gamma}+\frac{\epsilon^i\lambda_\alpha-\lambda_\beta}{\epsilon^{i+j}\lambda_\alpha-\lambda_\gamma}B^{i+j}_{\alpha,\gamma},$$

where $i+j\neq 0$ or $\alpha\neq\gamma$ and

$$B^i_{\alpha,\beta}A^{-i}_{\beta,\alpha}=\epsilon^i e_\alpha+(\epsilon^i\lambda_\alpha-\lambda_\beta)C_\alpha,$$

$$C_\alpha A^i_{\alpha,\beta}=\frac{1}{\epsilon^{-i}\lambda_\beta-\lambda_\alpha}A^i_{\alpha,\beta}+\frac{1}{\epsilon^i\lambda_\alpha-\lambda_\beta}B^i_{\alpha,\beta},$$

$$B^i_{\alpha,\beta}C_\beta = \frac{1}{\epsilon^{-i}\lambda_\beta - \lambda_\alpha}A^i_{\alpha,\beta} + \frac{1}{\epsilon^i\lambda_\alpha - \lambda_\beta}B^i_{\alpha,\beta}.$$

Here $\epsilon = \exp(2\pi i/k)$ and $\lambda_1, ..., \lambda_m, t_1, ..., t_m \in \mathbb{C}$ such that $(\lambda_\alpha)^k \neq (\lambda_\beta)^k$ for $\alpha \neq \beta$ and $t_\alpha \neq 0$. The elements

$$K_\alpha = t_\alpha C_\alpha + \sum_{(i,\beta)\neq(0,\alpha)} \frac{t_\beta}{\epsilon^i\lambda_\beta - \lambda_\alpha}(A^{-i}_{\alpha,\beta}B^i_{\beta,\alpha} - \epsilon^i e_\alpha)$$

satisfy the property $K_\alpha v = vK_\beta$ for all $v \in U(\mathcal{L})_{\alpha,\beta}$.

The corresponding operator R has the following components:

$$R_{\beta,\alpha}(x_\alpha) = \sum_{i\in\mathbb{Z}/k\mathbb{Z}} \frac{t_\alpha}{\epsilon^i\lambda_\alpha - \lambda_\beta}a^{-i}_{\beta,\alpha}x_\alpha b^i_{\alpha,\beta}$$

for $\alpha \neq \beta$ and

$$R_{\alpha,\alpha}(x_\alpha) = t_\alpha c_\alpha x_\alpha + \sum_{(i,\beta)\neq(0,\alpha)} \frac{t_\beta}{\epsilon^i\lambda_\beta - \lambda_\alpha}(a^{-i}_{\alpha,\beta}x_\alpha b^i_{\beta,\alpha} - \epsilon^i x_\alpha).$$

Here $A^i_{\alpha,\beta} \to a^i_{\alpha,\beta}$, $B^i_{\alpha,\beta} \to b^i_{\alpha,\beta}$ and $C_\alpha \to c_\alpha$ is a representation.

Let a, t be linear operators in some vector space. Assume that $a^k = 1$, $at = \epsilon ta$ and the operators $t - \lambda_\alpha$ are invertible for $1 \leq \alpha \leq m$. It is easy to check that the formulas

$$A^i_{\alpha,\beta} \to a^i, \qquad B^i_{\alpha,\beta} \to \frac{\epsilon^i t - \lambda_\beta}{t - \lambda_\alpha}a^i, \qquad C_\alpha \to \frac{1}{t - \lambda_\alpha}$$

define a representation of the algebra $U(\mathcal{L})$. Note that we do not assume that $t^k = 1$. We have only $at^k = t^k a$ which easily follows from the commutation relation between a and t.

Remark 1. If $m = 1$, then this is the Example 2.1. If $k = 1$, then this is the Example 3.1.

Remark 2. Since operator R depends linearly on $t_1, ..., t_m$, we obtain $m + 1$ pairwise compatible multiplications. One can check that these multiplications can be obtained using Theorem 1.1 from the case $m = 1$. We conjecture that the similar result holds for other Dynkin diagrams.

The cases corresponding to affine Dynkin diagrams of type D and E are treated similarly, but resulting formulas are more complicated. Note that classification of generic elements $1 \in \mathcal{L}$ up to choice of bases in the vector spaces $V_1, ..., V_r, W_1, ..., W_s$ is equivalent to classification of representations of a quiver corresponding to our affine Dynkin diagram with the same vector spaces. Therefore, we can apply known results about these representations (see [13, 14, 15]). Since the dimension of a representation is equal to mI, where I is the minimal positive imaginary root and our representation is generic, then it is isomorphic to a direct sum of m irreducible representations of dimension I. Therefore, $1 = e_1 + ... + e_m$, where $e_1, ..., e_m$ correspond to these representations. Taking the explicit form of these representations for affine Dynkin diagrams of the type D and E from [14, 15] and applying the scheme described above, one can obtain

explicit formulas for the corresponding PM-structures similarly to the case of the diagrams of the type A.

Conclusion

In this paper we have studied associative multiplications in a semi-simple associative algebra over $\mathbb{C}$ compatible with the usual one. It turned out that these multiplications are in one-to-one correspondence with representations of M-structures in the matrix case and PM-structures in the case of direct sum of several matrix algebras. These structures are differ from the Hopf algebras but in some features remind them. Namely, a PM-structure also contains two (associative) algebras $\mathcal{A}$ and $\mathcal{B}$, which are dual in some sense and satisfy certain compatibility conditions between them. Natural problem arises: to classify PM-structures for semi-simple algebras $\mathcal{A}$ and $\mathcal{B}$ (this is done in Section **4**) or, which is more difficult, to describe PM-structures if only one of these algebras is semi-simple (the case of commutative semi-simple $\mathcal{A}$ is treated in Subsection **2.3**).

Another interesting question is to investigate integrable systems corresponding to given representations of PM-structures. The problem here is to formulate properties of the integrable system in terms of algebraic properties of PM-algebra. It would be also interesting to study corresponding quantum integrable systems.

Note that our M and PM-structures are the particular cases of the following general situation. We have a linear space $\mathcal{L}$ with two subspaces $\mathcal{A}$ and $\mathcal{B}$ and a non-degenerate scalar product. The spaces $\mathcal{A}$ and $\mathcal{B}$ are associative algebras with common subalgebra $\mathcal{S} = \mathcal{A} \cap \mathcal{B}$. We assume that $\dim \mathcal{S} = \dim \mathcal{A} \cap \mathcal{B} = \dim \mathcal{L}/(\mathcal{A} + \mathcal{B})$ and our scalar product restricted on $\mathcal{A}$ and on $\mathcal{B}$ is zero. We have also a left action of $\mathcal{A}$ and a right action of $\mathcal{B}$ on $\mathcal{L}$ which commute with each other and invariant with respect to the scalar product (that is $(b_1b_2, v) = (b_1, b_2v)$, $(v, a_1a_2) = (va_1, a_2)$ for any $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and $v \in \mathcal{L}$). Finally, we assume that $\mathcal{A} \subset \mathcal{L}$ is a submodule with respect to the action of $\mathcal{A}$, where $\mathcal{A}$ acts by right multiplication and similar property is valid for $\mathcal{B}$. Now, if $\mathcal{S} = 0$, then we have the toy example from the Introduction, if $\mathcal{S} = \mathbb{C}$, then we have a weak M-structure and if $\mathcal{S}$ is a direct sum of m copies of $\mathbb{C}$, then we have a weak PM-structure of size m. It would be interesting to study and find possible applications of these structures for different $\mathcal{S}$.

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